The Rhyme of Reason

THE RHYME OF REASON

An Invitation to Accurate and Mature Thinking

BY

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To

GANZA

"She can be as wise as we,
And wiser when she wants to be."

PREFACE

THERE ARE two things about the structure of thinking that I have never been able to understand. The first is, why more people do not know something about it; and the second is, why more people do not give themselves the pleasure of playing with its patterns. I suppose the explanation is in part that most people do not realize that thinking has structure, that there is a rhyme to reasoning. As a matter of fact, it not only has structure, but this structure is in itself full of fun and fascination. In part, also, the explanation is that for so many the study of the structure of thinking has meant only pointless formalities and endless exercises.

The pattern of your thinking is as wonderful, as delicate, as intricate, as beautiful as an ivory carving that you would admire in a museum. Have you ever seen an ivory ball within a ball within a ball within a ball? The workmanship is superlative, the design is both satisfying and subtle. And the smallest part, the innermost ball, is as perfect and as interesting as the whole. In its dynamic phases the development of your thinking is as amazing as the growth of a flower.

You may have heard of the puritanical old lady who was told by her doctor that if she wanted to survive a serious illness she must submit to regular doses of sherry wine. Not a drop of alcohol had entered her house since she was old enough to do anything about it. She was in a desperate quandary. Was life worth while if borrowed from Old John Barleycorn? She finally decided that it was, so long as she did not *enjoy* her

doses! It is this type of puritanism that has done untold harm to studies of the structure of thinking. "Study it, but do not expect to *enjoy* it. logic must be difficult to be effective." You can almost hear the lugubrious tone of voice.

That, to my mind, is all wrong. If any subject is worth investigating, it is worth enjoying. And as for the structure of thinking, why people have not just gone out and enjoyed its patterns for themselves I do not know. We play games, we read detective stories, we absorb ourselves in all kinds of puzzles, we draw designs on scratch pads, we listen to music, we go to art exhibitions, we fool around in laboratories. There is not one of these activities that is not full to overflowing with pattern and form. Form has an inexhaustible appeal. And the quintessence of all forms is found in the patterns of our thinking. The patterns of thinking are as artistic as the patterns in a symphony or a painting.

And they are as effective as the structure of the George Washington Bridge. I can never understand why people will spend so much time and energy learning the structure of arithmetic and neglect completely the structure of the thinking that makes even arithmetic possible. Perhaps they want to get their money's worth, know that their change is correct, calculate the interest on an investment. But getting one's money's worth is far less a matter of arithmetic and interest rates than it is a matter of sound thinking in general. We do appreciate sound thinking when the doctor is trying to diagnose the symptoms of an illness. But we forget it again when we are well, and we go on leading muddled and empty lives when just a little application of the principles of valid thinking, an appreciation of its forms, would lead to far greater happiness and fullness of living.

I have tried to stress these things in writing this book. I have tried to apply thought patterns to the development of a philoso-

phy of life, to the achievement of mature thinking. I have brought in Sherlock Holmes stories, politics, yachting, picture puzzles, movie cameras, the growth of superstitions, brain-teasers, number systems, games of all sorts, and Mickey Mouse. I have even shown the reader a way to earn an easy million dollars by inventing a new game!

Of course, I shall be accused of popularizing. But I protest. I have omitted nothing from this study because it was difficult. The difficult I have tried to make less so by showing its excitement and significance. There are omitted, to be sure, certain items which would concern the graduate student of philosophy. But they are the cares of the professional and not of the amateur. Must the professional write only for professionals? Must the amateur be superficial? The amateur wants to learn and enjoy *and use*. And unless the professional can show that his subject is both significant and fascinating, as it must be if it is worth teaching, he had better close his shop. I shall be happy if I have been able here in small measure to show these things.

The fundamental tenet of this invitation to accurate and mature thinking is that one should gain an exciting appreciation of the technique of thinking rather than an apish facility in its technicalities. A book in this field should be a bid to a feast, should read like a novel . . . well, almost like a novel.

If this book makes any original contribution, it will be found in Part Four, in my attempt to describe the pattern of thinking as it advances from immaturity to maturity. The reader will see with one eye that I have been influenced by the logic of idealism here, but I hope he will recognize with the other that I disagree whole-heartedly with the metaphysical and dialectical sprees of Hegel and Gentile. I am convinced that the logic of maturing thought should have an integral place in any significant study of the structure of thinking, that it adds an essential vitality to

the subject. But I am aware that what I have done only scratches the surface of the problem. If the reader finds the general idea underlying Part IV half as significant as I secretly think it to be, I shall be satisfied.

The brain-teasers incorporated into the text, and the additional twenty-five collected in the Appendix, have been gathered over a period of years. Many have come from scattered and now forgotten sources, others are of my own invention out of patterns I have enjoyed in other problems. For some of the best I am indebted to William C. Hill of Springfield, with whom I have exchanged brain-teasers in friendly competition for many years.

The reader will also find in the Appendix a set of problems. I was undecided as to whether or not to include them in the book. There seemed to be some danger that they would be regarded as exercises, in the old-fashioned sense that has been so thoroughly exploded. But my wife, when she saw the problems, had the audacity to say that she thought them the best part of the book. As author I find it difficult to agree, of course. But in they go, partly for diversion and partly to provide opportunity for measuring one's understanding of thought structures. Some aim solely to illustrate thought patterns. Others try to show that concrete applications of the principles of accurate and mature thinking surround us in our daily life. The list of fallacies, for example, was gathered in just a couple of days' attention to newspapers, magazines, radio speeches, and conversations.

The Macmillan Company has kindly given me permission to use material from Castell's *A College Logic,* from Young's *Fundamental Concepts of Algebra and Geometry,* and from Ball's *Mathematical Recreations and Essays.* The Oxford Press has consented to the use of three long quotations from Jowett's

translation of Plato's *Apology*. And Charles Scribner's Sons has been good enough to allow me to take several items from Eaton's *General Logic* and Chapman and Henle's *The Fundamentals of Logic*. I am grateful for these permissions and have acknowledged them in the text. Several paragraphs in Chapter VII are taken from my article, "Your Nature and Mine," which appeared in the *Atlantic Monthly* in the spring of 1938.

Miss Juliet Fisher and Miss Margaret Lidgerwood have skilfully and patiently drawn the many diagrams scattered throughout the text. I appreciate particularly the labors of Professor Sterling Lamprecht and of my close friend and colleague Maurice Cramer, in reading the entire book in manuscript and giving many valuable suggestions. The touching picture of my wife propped up in a hospital bed, manuscript in lap and pencil in hand, has given me a bad reputation in a certain maternity hospital, but it should convey a small idea of her devotion to the chores of a professor's wife and of the difficulties under which she has contributed so ably and gracefully to this book. She titled it, by the way.

My greatest debt, however, is to the Mount Holyoke students of a half-dozen classes who have helped me in more ways than it is possible to relate to make interesting a study of the structure of thinking. Every page of this book carries me back to a classroom in which together we worked out apt illustrations and lucid comparisons. To these students my deepest thanks. This book should bear testimony that I learned more from them than they from me.

R. W. H.

TABLE OF CONTENTS

Part III

The Structure of Deductive Systems

Part IV

The Structure of Maturing Thought

The Rhyme of Reason

Logical consequences are the scarecrows of fools and the beacons of wise men.

Thomas Huxley

Introduction

THE LOGICIAN AND WHY HE USES SYMBOLS

IF YOU enjoy working out the strategy of games, tit-tat-toe or poker or chess; if you are interested in the frog who jumped up three feet and fell back two in getting out of a well, or in the fly buzzing between the noses of two approaching cyclists, or in the farmer who left land to his three sons; if you have been captivated by codes and ciphers or are interested in cross-word puzzles; if you like to fool around with numbers; if music appeals to you by the sense of form which it expresses—then you will enjoy logic. You ought to be warned, perhaps. Those who take up logic get glassy-eyed and absent-minded. They join a fanatical cult. But they have a good time. Theirs is one of the most durable, absorbing and inexpensive of pleasures. Logic is fun.

But it is more, too. You will probably not believe me when I say that logic is the most fundamentally useful and enriching of man's arts. To logic you owe not only the fact that you are alive, but your whole way of life. We live in a logical world. To act and think illogically is like trying to throw a ball with a dislocated shoulder—both futile and painful. Whatever our field of work or play we want to think accurately and maturely. Logic is the key to accuracy and maturity.

Logic is like marriage. Neither is appreciated until tried.

Both have bad reputations, but they seem to survive. The uninitiated will tell you not to forego your freedom, not to give hostages to fortune, not to enter upon the dullness of family life. A wife and children dull? They just do not know. There is nothing more important to our few years on this planet. The unknowing will tell you that logic is dull and impractical. They are equally blind. With both marriage and logic the unfortunate experience makes a louder noise than the more frequent and more peaceful good fortune. You cannot rescue a bachelor by telling him about the richness of married life. He will be caught, if he is not yet set in his ways, by more immediate considerations. And I cannot hope to catch you by telling you the fullness that logic will add to your life. But perhaps I can show you that it is fascinating. The bachelor gets married—and then watch him try to convert friends who have not taken the leap! You read this book—and then let me ask you if I was not right.

Logic Is the Study of the Structure of Thinking

There are certain experiences in life which are in themselves sufficient justification of our existence. Some of these are pleasures arising from pure sensation. The sight of a ski tip breaking softly into powder snow, of a white tennis ball against a clear blue sky at the moment before it is served across the net; the smell of salt water or of a ship chandler's; the sound of deep chord sequences in a Beethoven piano sonata; the touch of unvarnished wood smoothed by years of use, an old ax handle; the taste of Charles Lamb's roast pig. Life is too short for all that one would wish to sense.

There is another and quite different type of pleasure which enriches us, pleasure arising from a feeling for form. With one stroke of his brush an artist can paint a line which gives

keen enjoyment just by virtue of its shape. A Mozart piano sonata gives a sense of structure so delicate as to be almost incredible. Bach could weave seven voices into a pattern at once intricate and deeply satisfying. Men turn vase shapes on the potter's wheel, produce pages of type that are a sheer delight to the eye, execute figure-eights on ice, weave patterns into tapestries, write sonnets, beat the drum in the band. Art expresses structure, and in doing so gives pleasure.

But the realm of structure is not alone that of the artist. There is no content without a form in which it presents itself. Sometimes the form is superficial, as in the building of many houses; at other times it is distressingly inadequate, as in the program of the average movie theater. But form is always present. It has been said that form is a concern of the mind, while content is the concern of the senses. The senses give us a raw material, at times perfectly delightful just in itself, and it is the mind which either recognizes or imposes, as the case may be, the form involved. We recognize a certain form in nature and express it in the laws of physics. Henry Ford imposes a form on the raw material of various metals, and we recognize that form in the finished automobile. The form of the Model T marked an epoch in American history. The most dramatic example of man-made structure in nature is a bridge across a great river. Here the structural aspect is so important that one might say that it alone is responsible for the extraordinary achievement.

Not all structures are physical. There are many fascinating ones which have only ideal existence. Consider the structure of a game. Auction bridge has a form which differentiates it from contract. The form of each has fascinated thousands. Football, tiddledewinks, checkers—all have characteristic forms, and each can be classified with others of the same or similar patterns. Using the idea of structure one could invent a game.

As we shall see more in detail later, a good game has certain formal qualities. Combine them properly and you can make a fortune! Its elements must not be too complex, yet for the serious player there must be a wealth of possible combinations and strategies. The game must be well balanced. The ancient and honorable game of chess offers a structure perfect for those whose sense of form is highly developed. But more about games later.

The best example of a mental structure is arithmetic. Those who enjoy it from the point of view of pattern find it fascinating. The digits in any multiple of three add up to a multiple of three. The product of odd numbers is always odd. The intervals between the squares of whole numbers give an ascending series of consecutive odd numbers. There are countless patterns in arithmetic and many of the best mathematical puzzles are based on them. For example, what is the least number of weights necessary to weigh on a balance any integral number of pounds from 1 to 40? If you use only one side of the balance, you will need 1, 2, 4, 8, 16, and 32 lb. weights. If you may put weights on either side, you need only 1, 3, 9, and 27 lb. weights. The first is given by powers of 2, the second by powers of 3. Very neat. But why?

Logic is the study of the structure of thinking, the most universal and the most fundamental of structures. It is as important to intellectual transactions as arithmetic is to financial ones. Whenever you buy anything at a store, hand a bill to the clerk and get back silver in change, you are relying on a structure which makes one set of change "correct" and another "wrong." If it were not for arithmetic we should be reduced to a system of barter in our economic life. Exactly the same situation holds as regards thought transactions. When you argue a proposition with friend or enemy, bringing out relationships among ideas, you are relying on a logic structure which makes

one line of argument "valid" and another "fallacious." If it were not for logic we should be reduced to that mental bartering which characterized intellectual contacts among primitive men.

Thought, like anything else, possesses both structure and content. Neither has meaning without the other. If I stood before you and said, "Plum pudding," nothing more, your reply after an expectant silence would be, "Well, what about plum pudding?" The word, like the pudding, is rich in content, but as yet is part of no structure. If I write mysteriously on a piece of paper, "A loves B," I shall even more surely whet your curiosity. Who loves whom? I have stated a definite structural relation between two terms, but have omitted the vital content. We are going to examine the structure rather than the content of thinking, though we by no means wish to imply that structure alone is important.

There are occasions when content plays much the larger rôle. When the new father is told, "It's a boy," he is far more interested in what the doctor is saying than in the structure of the sentence. If we should encounter in a history book the statement, "Luther died a horrible death: he was excommunicated by a bull," we should certainly be more struck by the humor of the remark than by its structure. As a human being I should act with alacrity if my wife suddenly interrupted my writing to say, "Darling, I hate to disturb you, but there is a rattlesnake sliding along the back of your chair." My interest in the logic of her remark would be nil. But in our calmer and more rational moments structure plays so large a part with us that it may determine a whole way of life. It, too, has its moments: moments of fundamental significance, but also moments in which we thrill at the sheer neatness of a formal system newly discovered, ones in which we are amused by curious structural idiosyncrasies.

Perhaps you have stood prey to the fascination of a ship's engine room, caught and completely absorbed by the combination of power, delicacy, and precision. To contemplate the structure of thinking lends the same fascination. A machine exhibits extraordinary power under perfect control. Motions here, there, and everywhere that seem at first to be sheer confusion. Then one sees that this is geared to that, that the piston moves neatly forward and back while the shaft makes one ponderous revolution, that the eccentric over at the side performs its regular little task. One is left in breathless admiration of the whole, and entertained by favorite little details.

Microbes, Ferryboats and Ciphers

Have you heard of the fabulous microbes which multiply so rapidly that their numbers are doubled every minute? If you place two such microbes in a jar, they and their progeny will completely fill it at the end of an hour. How long would you have to wait before the jar is half-full? Of course you are expected to say "Thirty minutes," but the logic of the situation soon strikes and you revise your answer to "Fifty-nine." Brain teasers are full of logic. Some of the more difficult ones show clearly what logic can do.

Here is a problem the answer to which is amazingly simple when you see logically what is involved. Two ferryboats start towards one another simultaneously from opposite shores of a river. Traveling at constant speeds they meet 700 yards from one shore. They then pass one another, complete the crossings, and turn back without delay. The second time they meet they are 340 yards from the other shore. How wide is the river? You do not have to set up simultaneous equations in two unknowns in order to find the answer: all you have to do is discover one eloquent relation in the structure of the problem.

Can you find it? Most cannot. But this is it. The first time the boats meet they have together traveled the width of the river: the second time they meet they have together traveled three times the width of the river. Hence *700 yards* (the distance one boat goes before they first meet) equals one third of *the-width-of-the-river-plus-340-yards* (the distance the same boat goes in all).

$$700 = \tfrac{1}{3}\ (W + 340)$$
$$2100 = W + 340$$
$$1760 = W$$

And the river is a mile wide. It is amazing what logic can do. Reason is a powerful instrument.

Consider another illustration. During the World War our Intelligence Service was faced with the important task of deciphering enemy ciphers received by wireless. The ciphers were complex and provide an interesting study in themselves. Suppose for a moment that you were confronted with a cryptogram. Here is a simple one that contains no military secrets:

BABCEOR RUT ISACH SY RUT LTFCI,
UGJBE ISACH ORKCFTO CE FBCE.

How would you proceed? Obviously you would not make substitutions at random in the desperate hope of uncovering the meaning. *You would reason.* And in reasoning you would employ *logic*. You would see immediately that either S or Y, and either C or E, is a vowel. If Y were a vowel it would have to stand for either *o* or *e* (last letter in a two-letter word). But these two vowels usually occur often, while Y appears only once. Let us say, then, that S is a vowel. If E stands for a vowel, then it must stand for one that often ends words. But since it cannot stand for itself and *e* is the only letter that frequently ends words, it seems more probable that C stands for a vowel. If, then, S and C are both vowels, the word "ISACH"

suggests that I and A and H must stand for consonants. In this case, B is a vowel (see BABCEOR). Proceeding in an orderly and rational manner, you are well on the way to the solution of this cryptogram. It is quite probable, for example, that SY stands for *of*. Finish it for yourself, and notice that your thinking has structure. Whenever you use such words as "if," "then," "either," "or," "since," the reasoning you are employing will be apparent.

While we are on the subject of ciphers, it is interesting to note that one employs reason not only in deciphering messages, but also in inventing the original cipher. It is a matter of history that in the World War no country was able to construct a cipher which its enemy could not decipher. The explanation of this is easy. Man is a rational animal, at least in the Intelligence Service he is, and the same reasoning employed in making the message secret could be used to break it. Codes, of course, are different. One may use a code book, and if the recipient does not own a copy no amount of work will bring out the meaning of the message, *because it is not based on a rational system.* But even in such cases the rational approach is helpful. There is the famous case of a Mexican wireless station broadcasting to Berlin during the World War, which suddenly changed its code to one which no amount of labor made intelligible. But it was full of *numbers*. References to pages and lines of a book? The change was so sudden that our decoders were reasonably certain that no special code book was employed. The Mexicans must be using some ordinary book. What book would surely be found both in Berlin and in Mexico City—and would have all necessary words? A German-Mexican dictionary? An examination of such dictionaries gave the key in short order!

Logicians and Purple Cows: the Difference between Truth and Validity

Is it *true* that I should undergo an operation? Is it *true* that he is an honest public servant? Is it *true* that hydrogen and oxygen combine to form water? Is it *true* that the square root of 169 is 13? Thinking is not intended to be carried on in a vacuum. It applies to a real world, and is true or false with regard to that world.

We have said that all thinking has structure and content. *Truth* describes content: *validity* describes structure. The logician is concerned exclusively with the latter. It is difficult at first to get accustomed to the idea that he does not deal in truth values. But the problem of truth is quite separate. By means of faulty logic we can arrive at true conclusions:

8 is greater than 4
and 12 is greater than 4
Therefore, 12 is greater than 8.

The structure of this argument is bad, but the conclusion is true. It is true that some men smoke pipes, but if we argue this truth on the ground that all men seek pleasure and that pipe-smoking is a pleasure we shall be arguing invalidly.

This is not to say that the question of truth is unimportant. If we start from untrue premises we may proceed with perfect accuracy of argument and arrive at a falsehood:

4 is greater than 12
and 8 is greater than 4
Therefore, 8 is greater than 12.

There is no profit in this, though we shall see in a moment that if the structure of this argument is expressed in symbols it will be found valid. The best logician in the world will get into

trouble if one of the premises of his action asserts that all men can be trusted.

Happily for mankind we are often illogical in our thinking yet right in our conclusions. But the situation is not desirable. Our chances of arriving at the truth are better if our thinking is accurate. *Truth* describes the relation between a concrete statement and the reality to which it refers. It is the problem of the artist, the scientist, the mathematician or the philosopher. *Validity* describes that structural relation between two or more propositions which allows us to say that *if* one or more are true, *then* another or others must also be true. If *cows are purple* and *this is a cow,* then *this is purple.* It makes not a whit of difference *to the logician* whether or not there *are* purple cows, or whether this is a cow or a horse. But if *this* and *this* are accepted, then *that* must be accepted also, solely on the basis of thought structure. The logician asks: What is the relation between thoughts such that certain ones lead necessarily to others? It is trite to remark that "if" is a little word with a big meaning. But the logician must never forget it, for it is the preface which must be attached to every statement he makes.

The logician uses symbols. Symbols allow him to remove content, leaving only structure. If I say, "A is greater than B," I speak neither a truth nor a falsehood. The problem of truth cannot even arise until it is known what "A" and "B" stand for. But "A is greater than B" has a highly significant structure.

One Reason Why the Logician Uses Symbols: Abstracting Form from Content to Find Structure

If you are paid two dollars an hour for an eight-hour job you do not insist that the paymaster hand you eight piles of money. You simply multiply two and eight. *You use symbols.*

If you know that two brothers each own eight houses, you do not have to go around counting all of the houses to know that together they own sixteen. Our lives would be enormously complicated if we did not rely on the structure of arithmetic. Many things we should not have time to do. Imagine trying to carry on a business. Or finding the national debt! Arithmetic is powerful because it expresses in symbols a structure which is the same regardless of the objects being counted. Given a concrete problem, we can express it in symbols, then manipulate the symbols until we have our answer, and interpret the answer in terms of things.

We can have exactly this same power in manipulating our thinking. I read in the newspaper the following argument: "If X——— is an honest politician he will have reduced taxes. He has reduced taxes, and therefore he is an honest politician." Most of us would let that argument by as valid. But consider another: If I eat too much ice cream I will be sick; I am sick, therefore I ate too much ice cream. One of these arguments deals with politicians and taxes, the other with eaters and ice cream. Their contents are quite different. *But their structures are exactly the same.* Observe that each contains two premises: the first premise states that if a certain proposition is true another will also be, the second states that the other proposition *is* true. Both attempt from these similar premises to draw necessary conclusions. If we let "P" stand for the first proposition and "Q" for the second, we may symbolize the structure by saying: "If P is true, then Q is true: and Q is true. Therefore P is true." We may further symbolize by using "$\supset$" for "if . . . then" and "$\therefore$" for "therefore." The complete symbolization then becomes:

$$\begin{array}{rl} & P \supset Q \\ & \underline{Q} \\ \therefore & P \end{array}$$

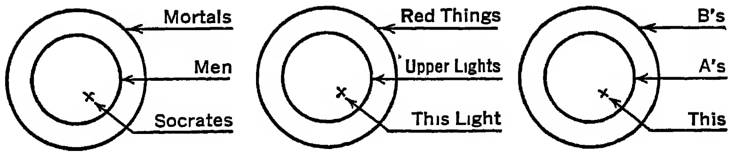
We are now expressing pure structure, the structure of one type of argument: and it is clear that the structure is faulty. No matter what you are arguing about, if you use this form you are arguing invalidly. The fault in your argument depends in no way upon the facts of the case, but only upon the relations of your propositions one to another. This will sound altogether obvious now that we have seen the symbols, but clothe the structure in a familiar content and you may be fooled. I have seen many intelligent people accept the argument about the honest politician without a murmur, and its type appears in political speeches *ad nauseam*. You will have no trouble finding one in your newspaper.

Similar Structures in Similar Arguments

Every logic book brings in the famous case of Socrates and his mortality. All men are mortal, and Socrates is a man: therefore, Socrates is mortal. Another argument with the same structure would be one which a color-blind person might use. In traffic lights the upper light is always red, and this light is the upper of the two: therefore, it is a red light. In the argument about Socrates neither premise will be questioned; in the other, there is room for doubt. But both have the same structure, one which is often and conveniently shown by an analysis of the classes involved:

All A's are B's.	A = *man*, or *upper light*
This is an A.	This = *Socrates*, or *this light*
Therefore, this is a B.	B = *mortal*, or *red thing*

Another way of symbolizing the structure of these two arguments is to draw circles. Circles make the similarity of structure graphic:



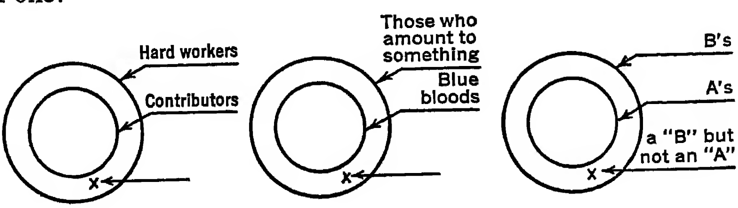
From the drawing it is obvious that if the first class mentioned is entirely included within the second class, and the individual object is a member of the first class, then the individual object must also be a member of the second class.

You may object that we do not need to draw circles, or even to analyze structure, to see the validity of these arguments. True. But not all of the structures of argument are as clear cut as this. If you believe that all men who have made great contributions to human progress have been hard workers, the chance is very good that you will dismiss as lazy the person who has made no contribution. It is not infrequent for one of the social aristocracy to argue that those with blue blood in their veins amount to something, and that therefore the lowly clerk who wants to marry his daughter will never amount to anything. Here the structural analysis is more serviceable:

All A's are B's
This is not an A

Therefore, this is not a B.

If we draw circles, showing that all A's are included in the class of B's, it becomes clear that "this" does not necessarily fall outside of the B circle simply because it is excluded from the A one:



Sometimes it is the language rather than the inherent structure that introduces difficulties. Nine people out of ten, even on examinations, say that the following argument is correct logically:

Only manuscripts accompanied by return postage were returned.
This manuscript was accompanied by return postage.
Therefore, this manuscript was returned.

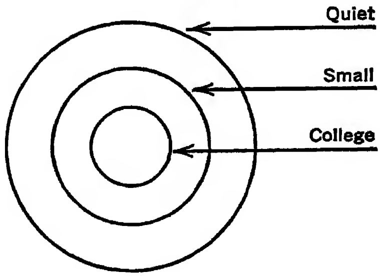
But it was not. Why? Draw your circles and see, but be careful how you draw them!

Arguments of Different Types May Have Similar Structures, Too

So far we have been illustrating similarities of structure in arguments of the same type. In the last few paragraphs, for example, we have been dealing with relationships of one class to another and of individuals to classes. We have been able to draw illustrative circles. But similarities of structure also occur among arguments of different types. I might argue validly:

Small towns are quiet.
College towns are small towns.
Therefore, college towns are quiet.

This is easily symbolized by three circles arranged roughly like the circles on a target.



But the following argument cannot be handled in the same way:

Mary is taller than John.
James is taller than Mary.
Therefore, James is taller than John.

Yet it is plain that there is a similarity of structure between the two arguments. They both involve what we shall later recognize as *transitive* relations between the terms involved. The first is the relationship of class inclusion and the second is the relationship of the comparative. Their underlying structures might be symbolized in the following manner:

$$\begin{array}{llll} & A & R_t & B \\ & C & R_t & A \\ \hline \therefore & C & R_t & B \end{array}$$

Again, this time in a different manner, we have extracted the pure form of the argument from its content. It makes no difference what A and B and C stand for, so long as the relationship between them is transitive. Wherever this is the case the argument will be valid. There are a large number of transitive relations. Many are spatial, as for example, "is north of." New York is north of Philadelphia, and Philadelphia is north of Washington: therefore, New York is north of Washington. Others are temporal: "is earlier than," is one.

We now understand one reason for the employment of symbols in logic. Symbols express generalizations, abstract the form from the content. A's and B's stand for *any* classes of things, P's and Q's for *any* propositions, and R_t stands for *any* transitive relation.

A Second Reason Why the Logician Uses Symbols: Words Are Slippery Customers

Words play extraordinary tricks. That is another of the reasons why it is essential in logic to employ symbols. Similarity in wording can often be completely misleading if one attempts to infer similarity of structure from it.

A desk is a piece of furniture.
∴ A brown desk is a brown piece of furniture.

All right. The following argument, however, looks superficially much the same yet is palpably bad:

An artist is a man.
∴ A good artist is a good man.

The trouble arises in the fact that a word, "good," is employed in two different senses. If our language were such that each word were a symbol for a single idea many such fallacies would be avoided. These fallacies are known as *fallacies of equivocation.* Sometimes the alteration in structure produced by a single substitution of wording can lead to ludicrous results. We accept the argument about the relative heights of Mary, James and John. But substitute another relation for "is taller than" and see what you get:

Mary loves John.
James loves Mary.

Therefore, James loves John.

"Loves" is anything but a transitive relation! The similarity here is entirely superficial. From these two examples it is obvious that it is one job of logic to improve on ordinary word use.

Words, words, words. Now you understand them, now you don't. Step right up, ladies and gentlemen. Here we have the startlingly ambiguous advertisement: "Wanted: a chair for a lady with a cane seat." And there is the oracle whose pronouncement is open to misinterpretation: "Alive is the duke that Henry will depose." Poetry often involves double meanings. Could a logic book be written which did not mention, "All that glisters is not gold"? The Old Testament provides a fertile field: "And he said, Saddle me the ass. And they saddled *him*."

There are times when the substitution of a single word for another will produce the most startling changes in structure. We may place side by side the two following arguments, which from the point of view of their superficial structure are as similar as any two could possibly be:

Small towns are quiet.	Small towns are numerous.
College towns are small.	College towns are small.
Therefore, college towns are quiet.	Therefore, college towns are numerous.

The first is valid: the second is not. And the sole difference is that between the structure implied in the word "quiet" and that implied in the word "numerous." The former describes *each individual* small town as being quiet in character: the second refers to a characteristic of the class of small towns *as a class*. Only a class can be numerous: an individual may not be referred to as such. Hence two different kinds of words, the difference between which we must bear in mind if we are to think accurately. Expressed in logical symbols the difference becomes quite clear:

Small towns are quiet.	$A < B$	A is included in B
Small towns are numerous.	$A \in B$	A is a member of B

Often, of course, the distinction between words which describe individuals and words which describe classes is unmistakable. If I should try to argue:

> American Indians are disappearing.
> He is an American Indian.
> ───────────────
> Therefore, he is disappearing.

you would hardly take me seriously.

Two Common Fallacies Arising from the Misuse of Words, and an Oddity of Grammar

In confusing words which describe individuals as such and words which describe classes as such, two fallacies are often committed. They occur when a single word offers both possibilities. It would probably be a serious mistake to speak of a member of a dense crowd as being himself dense. You would then commit the fallacy of *division*, or "dividing" a word used for a class so that it applies to each individual in the class. A similar fallacy may be committed in the other direction. Any athletic coach will tell you that a team made up of good players is not necessarily a good team. To maintain that an All-Star team is the best team is to commit the fallacy of *composition*, or using "compositely" a word intended to apply only to individuals. The error involved is not always clear. Here is an eloquent example of this fallacy: "The men of the army consist of privates, non-commissioned officers and officers of all ranks. Those who fight in the front-line trenches are men of the army; therefore they consist of privates, non-commissioned officers and officers of all ranks."

Sometimes the structure of grammar will suggest an argument whose logic leaves much to be desired. There is probably no better example of this than the following, taken from

Castell:[1] "The only proof that a thing is visible is that people see it. The only proof that a sound is audible is that people hear it. The only proof that a thing is likeable is that people like it. The only proof, therefore, that a thing is desirable is that people desire it." In the first three instances the suffix carries the meaning, "capable of being," whereas in the last the meaning is "worthy of being." But surely an uncritical reading of that argument would show the fallacy to few of us. Try it on some one.

The Growth of Language and the Ambiguities Arising Therefrom

Words are slippery. Their meanings change with usage, and sometimes the change is great. The original meaning of "treacle" was *"pertaining to a wild beast!"* Later it meant the remedy for the wound given by a wild beast; then any remedy; then a remedy in the form of a syrup; and to-day we think of molasses when the word appears. Chaucer's phrase, "the treacle of Christ," is thoroughly misleading until you know this history. Not only do single words suffer alterations in meaning, but often they are taxed with shadings of meaning until the same word may have a variety of definitions. One of the most serious offenders is the word, "good."

Perhaps the worst offender is the seemingly innocent word, "is," which occurs so frequently in discourse. Consider these six examples:

If I say, "Logic is dangerous," I assert the *predicate* "dangerous" of the single subject, "logic." Rx

If I say, "All logicians are menaces," I assert that the *class* of logicians *is included in* the class of menaces. A < B

[1] *A College Logic* (Macmillan), p. 46.

If I say, "The writer of this book is a menace," I assert that I am *a member of the class of* menaces. $A \in B$. We have already noted the confusion that arises from confusing this with the "is" of class inclusion.

If I say, "To write is obligatory," I assert the relation of *implication* between two propositions, "I write" and "I am fulfilling an obligation": *if* one writes, *then* one is fulfilling an obligation. $P \supset Q$

If I say, "The writer of this book is myself," I assert an *identity* between the writer of this book and myself. I mean that either side of the identity may be substituted for the other in discourse. They are the same. $A = B$

If I say, "Truth is Beauty," I assert an *equivalence* that may hold in art but which certainly does not apply to discourses on logic. I mean here that two propositons, "This is true" and "This is beautiful" mutually imply one another. *If* this is beautiful, *then* it is true: and *if* it is true, *then* it is beautiful. But the two are not identical. $A \equiv B$

The logician must give each of these meanings a different symbol, for they signify different underlying structures and involve different manipulations, as we shall see more clearly later.

Of the six, *identity* is pertinent to our present discussion of the use of words because by means of it new words come into usage and older words retain their meaning. Words are symbols, extraordinarily useful ones. If we had to use an individually different designation for each individual egg, think how complicated a farmer's job would be. Primitive language was amazingly crude. At first we could only point at objects and grunt—sign language. Gradually the grunt for "Take it away" was distinguished from the grunt for "Bring it here." Then men discovered that they could make sounds stand for objects, and they no longer needed to have a fire around to point to when they wanted to refer to one. Early languages were sheer matters of custom and habit: if an influential man in

the tribe called some new-fangled device a "glorb" the others followed suit. The tribe across the river might call it a "brolg," but for a while that did not make much difference.

After men had advanced to a more sophisticated attitude toward language they got self-conscious and the dictionary-makers and grammarians came into existence. Definitions became the weapons of the academically-minded and a source of woe to the schoolboy. Dictionaries have contributed much to clarity of language, but even they must ultimately follow custom. They could warn people in the boldest print that "fine" means small, but if new generations persisted in having a "fine time" the dictionary could do nothing but add a new definition to the old one.

Language is enriched and kept vital by this growth. Slang comes and goes and the best is retained by a sort of survival of the most eloquent. Who would have wanted to miss "twerp" or "bender"? We seek force and beauty in language, but we also seek understanding. All too often a long bull-session will end with, "Oh well, if *that* is what you mean by 'socialism' then I agree with you," or "We're saying the same thing, but you and I mean different things when we speak of 'God.' " Up to a certain point, the more words at our disposal the more accurately we can convey our meanings. How do they enter the language?

New Words Come into the Language and Have to Be Defined

You and I are standing on a wharf watching a boat go by with a girl at the helm. If I say, "Isn't she a beauty!," there is very large ground for misunderstanding even if you admire her "shape" or "lines" in reply. If I say, "Isn't that boat a beauty!," we shall have a more secure common ground for

discussion. But there are lots of boats in the harbor. You may have an eye for motorboats rather than sailboats. I would do better to refer to the "sloop," and better still to mention the "Friendship sloop." By what right do I use the word, "sloop"? By right of the fact that we have agreed to *identify* "sloop" with "a vessel having one mast and a fore-and-aft rig consisting of at least a mainsail and jib." In one sense a new word is a shorthand; in another it is a labor-saving device. Suppose there were no distinction in language between steamers, tugs, ferries, destroyers, punts, canoes, catboats, submarines, sampans, etc. Suppose we could refer to them only as "boats." It would have to be "this boat" or "that boat" or "the boat you were on yesterday" or "the boat just entering the breakwater" or "the boat with the orange funnels." The limitations are obvious, especially if you are trying to carry on a maritime conversation in Kansas. Of course there is a limit in the other direction. If we make up too many words or allow them to become too complex, advantages disappear. The German word *Reisegepacktsversicherungsacktiengesellschaften* carries a good thing too far.

We are often, embarrassingly often, called upon to clarify the meanings of the words we use. A good definition meets certain requirements. They may seem obvious, but it is surprising how often they are disregarded. In a recent book whose author shall be nameless I encountered the following: "A rational life is one lived according to the dictates of reason." How much does that advance our knowledge? The word or phrase being defined must not be repeated in the definition. The virtue of a new word is that it is defined in terms of *different* ones with which we are already acquainted. A new thing or idea is identified with a group of things or combination of ideas previously describable only by long explanatory phrases. It will also be apparent that a good definition is positive rather than negative. "A schooner is not a horse" tells us very little. In

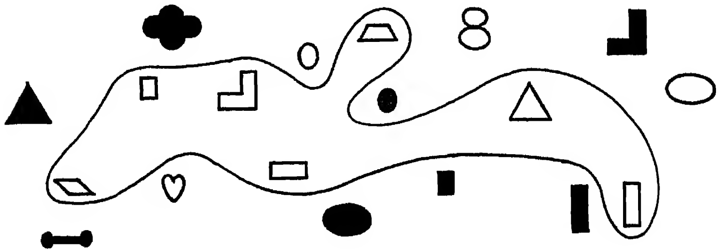
defining words avoid negatives whenever possible, for they probably indicate fuzzy thinking.

The Function of a Definition, and Its Elements

Man is a rational animal. A rectangle is a quadrilateral whose angles are right angles. All definitions contain two elements, usually referred to as *genus* and *differentia*. The thing or idea being defined is first identified with a large group of things or ideas with which we are already familiar. Man is an animal. A rectangle is a quadrilateral. A sloop is a vessel. This is its "genus." But what kind of an animal is a man? What kind of a quadrilateral is a rectangle? What kind of a vessel is a sloop? That is where the differentiae come in. "Rational" is the differentia which distinguishes man from among the larger group of animals. Similarly with "whose angles are right angles." When logic was more metaphysical than at present it used to be said that the differentia must include an essential property of that which is being defined. Is it of the essence of man to be rational? That is certainly a question for the philosopher or psychiatrist, not the logician. From the point of view of the function which a definition performs, man might equally well be defined as a laughing animal. But surely it is not of the essence of man to laugh.

Suffice it to say, once obvious conditions are met, that we may ask two important questions of any definition. If they are satisfactorily answered, we may accept the definition. (1) Does the definition exclude all instances which it is intended shall be excluded? (2) Does the definition include all instances which it is intended shall be included? It is difficult to construct a definition of "democracy" which will exclude Soviet Russia, yet we mean by the word something which does exclude Russia. It is equally difficult to find a definition of "life" which shall

include all phenomena which we wish to bring under that term. One might symbolize the function of a definition in the following manner:



We want to describe all of the figures inside the line, and only these, by using a single word. Having no word as yet, we make up "gratches" and our definition of gratches will be a good one if it (1) fits every figure within the line and (2) excludes every other figure. We might continue to refer to "all of the figures on page 24, between lines 3 and 4, which are both hollow and have straight edges." But "gratches" is more convenient. The relation of identity, and hence the function of a definition, involves simply the substitution of a word for an unwieldy phrase.

A Third Reason Why the Logician Uses Symbols: Emotion and the Fallacies for Which It Is Responsible

At a recent patriotic dinner in New York a man got up to argue for a Big Navy program. It was foolish to suppose, he said, that if we had a big navy we should be paving the way for war. Would any one argue that an increase in the fire-fighting apparatus in New York City would lead to an increase in fires? Why, then, should a large navy lead to war? One can easily picture the atmosphere: ready applause, patriotic senti-

ments, American flags, red-white-and-blue decorations. Were many of his hearers in a state of mind to examine this man's argument critically? Probably not. Yet the evidence brought forward about fire-fighting was about as irrelevant to his point as it could have been. It would have been just as logical to mention that his name was Augustus, which it was not. Beware of the speaker who employs the *analogy*. It is highly untrustworthy.

The frequency and force with which irrelevant evidence may be brought into an argument are amazing. It is usually based on emotion, an appeal to the sentiments of the hearers. Emotions distort thinking. Hence another of the functions of logic, to remove the emotional element. Nothing could have less emotion attached to it than a logician's symbol! If a man offered to sell you a wonderful insurance policy provided you would pay him a penny the first day, two the second, four the third, eight the fourth, and so on for only a month, you would know enough (I hope, for your sake!) not to be taken in by the fact that you were dealing only in *pennies* and paying for only *a month*. You would sit down and figure it out with pencil and paper. If juries and political audiences would do more of this sort of thing we should be living in a better world. You may rely on arithmetic, and you may rely on logic. Consider the many forms which the fallacy of irrelevant evidence can take.

We are asked to keep out of foreign entanglements because of George Washington; to vote Republican because of Abraham Lincoln or Democratic because of Thomas Jefferson; to disparage a college education because of Henry Ford; to retain the jury system because of its venerable age. *Appeal to great names and great institutions!* But would Washington keep us out of foreign entanglements to-day? The question is certainly an open one. Would Lincoln or Jefferson vote with his party

to-day? Is Ford an authority on education? Is the jury system desirable just because it is old?

Sometimes the appeal is to *pity*. We all know the type of person who can turn on the tears at will. We know the attorney who can make the jury sniffle. The child who cries for candy knows what he is doing, and so does the professional beggar. But sympathy for the prisoner or defendant does not change the question of guilt or innocence. The satire in Gilbert and Sullivan's *Trial by Jury* goes all too deep. Sometimes the appeal is to the *Old Adam* in us. We all know, from the movies at least, of the care taken by the female defendant to dress attractively and cross her legs at just the proper angle *if* the jury is male. You cannot meet that situation with a pencil and paper, at least not properly, but you can be logical about it. Similar to this is the appeal to the *prejudices* of a definite group. Flag-waving is such an appeal, and usually has nothing to do with the case. Appeals to race prejudice are also common.

The classic example of irrelevant evidence is the "Friends, Romans, Countrymen" speech of Marc Antony, so clever that it runs almost the entire gamut of prejudices, ending with one of the strongest, the money appeal. What is a bribe but an example of extraordinarily poor logic? But worse even than this is the "argument" of the racketeer or the rubber-hose artist: "Do this, or else. . . ." The appeal to *force* is the least rational of them all. One of the most successful of the irrelevant tricks of debate is the appeal to the ignorance of one's antagonist. "There is no God—and I defy any one to prove me wrong." If you cannot prove your own case, it is tempting to challenge your opponent to prove his. It may swing an audience or a jury, but it does not in itself strengthen your argument.

There are other tricks and fallacies in argumentation, all made possible by the fact that we can be so influenced by the material being presented as to neglect to observe carefully the

structure involved. The police inspector in detective stories is made to look like a monkey because of *hasty generalization.* We have all encountered this argument: "This employer is dishonest, that employer is dishonest, the other employer is dishonest: all employers are dishonest." How easy it is to describe Frenchmen after a week in Paris! How easy, too, to commit the similar fallacy of *neglected aspect.* We are ready to believe what we want to believe: we read the newspaper which tells us what we want to know. One editorial brings forward statistics showing the detrimental effect of monopoly by power trusts: another martials the facts against government ownership. An argument rarely tells the whole story impartially. If we are to think logically we must be on our guard. Unwisely *broad generalizations* offer another source of error. It may have been sheer crudity to believe that all Russians wear beards and carry bombs; but not many years ago the average cartoonist foisted this idea on a multitude. It is equally dangerous, though much more sophisticated, to argue that any man who kills another is committing a crime.

There is one more important type of fallacy into which we can be swept if we do not watch ourselves very carefully. It has the elegant name, *Post hoc ergo propter hoc,* which being translated means *coming after therefore caused by.* "As soon as he entered the house the trouble began." But was he the cause? A sudden increase in armaments is a fairly reliable sign of an approaching war. From this many will argue that it is the cause of war, whereas it is more probable that both are caused by something more fundamental. One might as well argue that wedding gifts caused the wedding. Are high prices the cause or the result of prosperity? The question of causes is always an intricate one, to be decided on the merits of each individual case. That one event precedes another is no good argument that it caused it.

Hence the Importance of a Study of Structure

These are the adversities of rational thinking, and these the reasons for the employment of symbols. Symbols are at least impersonal. One cannot get emotionally wrought up over A's and B's and P's and Q's. One sees the argument in its true form, and often it ceases to be recognizable as an argument because it is sheer appeal to sentiment or tradition or both. Human nature being what it is, our thinking is far more often governed by emotion than by reason. Aristotle defined man as a rational animal, but one good look at the contemporary scene makes this appear to be uncomfortably grim humor. Now there is nothing wrong with the human emotions. Let's not get puritanical about this. But they do need guidance. Logic is a constant struggle against the weaknesses of human nature.

We might as well face it. It is just not human nature to think clearly. Accurate thinking requires attention to structure, and attention to structure requires effort. We are lazy creatures. We fall easily into that which is sham and that which is ugly and that which is narrow. It is so much easier to think the way we want to think than the way we should think. The child cries for the candy that will make it sick, sees only the pleasure of the moment, sees things as it would like them to be. How much better to look ahead to to-morrow, to find the enduring satisfactions of life, to see things as they are. Logic helps us to grow up.

PART ONE

The Structure of an Argument

Chapter One

WHAT'S IN A PROPOSITION?

WHEN Adam first saw Eve in the Garden of Eden he probably did not tip his hat and say, "Madam, I'm Adam." There are some who think he must have, for they find no other way of explaining those fanatics among Adam's progeny whose hobby it is to play around with alphabetical structures. One of these fanatics is the palindrome artist, who is always scribbling on little bits of paper in the hope of finding meaningful sentences, like that attributed to Adam, in which the letters are symmetrically arranged either side of the middle. Another of his prize products, this time having no reference to Eve, we hope, is: "Was it a cat I saw?"

Then there are the anagram fiends who take groups of letters and rearrange them with new meanings. They are thrilled to discover a word like "united," which needs only the switching of two letters to produce another of precisely the opposite meaning, "untied." They are overjoyed to discover that "Best in prayer" becomes "Presbyterian." In the higher reaches of their art they find that the letters in "The midnight ride of Paul Revere" can be transposed into "Rider gave hint of peril due"; that "Washington crossing the Delaware" may be either "He saw the ragged Continentals row" or "A hard howling tossing water scene." They even bring history up to date and

transpose "Chancellor Adolph Hitler" into "Lord of all the Reich clan."

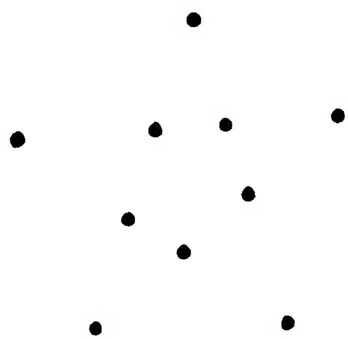
These are just a few of the discoveries that lie ahead when you begin to inquire into the structure of language. Though alphabetical structures are highly entertaining, and sometimes so amazing that they seem to have been planned in advance, they are of no particular significance. Certainly they are not of logical significance. They do not give the structure of thinking as such.

When men began to concern themselves with logical structure they hit upon an analysis of the single proposition with which we are made familiar in school, the subject-predicate analysis. Realizing that every proposition contains both a subject and a predicate, they concluded that these two were the most fundamental logical elements. In every proposition a subject is qualified by its predicate; for all the formula is *S is P*. For reasons which it is unnecessary to divulge at the moment, it was later seen that this type of analysis, while grammatically correct, was logically awkward. It did not allow expression of the refinements of structure which we actually find in our thinking. While logically more significant than the alphabetical analysis, it still left something to be desired.

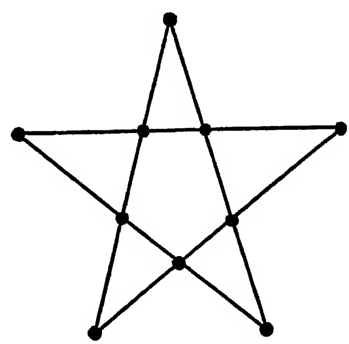
What in General Is a Structure?

Perhaps the best way to discover the structure of thinking is to examine structure in general, and see if we can find anything of large significance. Among structures, one of the most familiar is space. Some of the best puzzles and games are based on structural aspects in space. Have you ever been asked to arrange ten pennies in five rows of four each? There is significance just in the question itself. Notice that we are asked to place *things* in a certain *relationship*. But there is also an

interesting aspect of the solution. If I simply arrange the pennies thus:



the relationship is not entirely clear. But if I draw five lines you see immediately the plan on which I worked:



The relationship is definite and concrete, geometric in origin. Other puzzles are based on other relationships. Can you place jacks, queens, kings, and aces of four suits in four rows and columns so that no two cards of the same suit or value will be found in the same row or column? The formal aspect of this problem is quite clear. Can you place eight queens on a chess board so that no one can capture another? It will take some

good clear thinking to work this out and, as regards form, the solution is itself something of a puzzle!

Among the most useful of all of the structures known to man, and to some the most fascinating, is that of arithmetic. And here, again, we have *things* in *relationships*. The "things," or course, are the numbers. In this case it might with justice be said that the things would be of no significance were it not for the *relationships* among them. The three most familiar relationships are "is equal to," "is less than," and "is greater than":

$$8 + 4 = 12$$
$$4 < 8$$
$$8 > 4$$

Other familiar relations operate on numbers to give new numbers, thus adding to the complexity but also the usefulness of the system. Thus:

8	*plus*	4 gives the new number	12
8	*minus*	4 gives the new number	4
8	*times*	4 gives the new number	32
8	*divided by*	4 gives the new number	2

Less familiar relations in arithmetic are "aliquot part of" and "incommensurable with." All puzzles of the "Take any number from one to ten. . . ." variety are based on special relationships between the numbers of arithmetic. Multiply any number by three, divide the result by the number, then add seven and your answer will always be ten, to give a simple example. You will see why almost immediately.

In all of these cases of pennies, cards, chessmen, and numbers we notice a characteristic common to the systems into which they fall. All of the systems consist of *things in relationships.* The important point at the moment is that this is also true of propositions in our thinking. All propositions may be analyzed

into *terms* in *relationships*. *Terms alone*, "John" or "letter," tell us nothing. Relations alone, such as "writes" or "receives," also leave something to be desired. But put them together and you get complete propositions, "John receives a letter" or "John writes a letter."

And What in Particular Is the Logical Structure of a Proposition?

The term "penny" belongs to more than one structure. We think of it first as part of our monetary system, and only secondarily as providing interesting spatial relationships. In the puzzle nickels or dimes would do as well. Playing cards belong to any number of systems: poker, cribbage, bridge, etc. "John," "letter," "writes" and "receives" belong (as the others do also, of course) to the system of grammar. They also belong to other systems. Systems overlap.

The next part of our problem is to discover the characteristics of *logical* elements. What is the term *in logic?* What are the characteristics of *logical* relationships. We must be discriminating. Consider the relationship "and" in the two following examples:

8 *and* 12 is greater than 4
8 *and* 12 are greater than 4

Structurally the two *look* much alike. But actually the first is a single proposition while the second is *two* propositions. The first would be expressed symbolically as $8 + 12 > 4$, and the second would be expressed as $8 > 4$ and $12 > 4$. In other words, the first "and" is an arithmetic relationship ($+$) while the second is a *logical* relationship. We shall want to be certain that in analyzing propositions in the search for the structure of thinking we analyze them logically rather than in some other way. The grammarian's analysis of a proposition, for ex-

ample, leads in the direction of all of the intricacies of syntax with which we are made so painfully familiar in school: noun, verb, adjective, adverb, preposition, and so on. We are seeking something quite different.

What structure are we seeking? It is difficult to say. If I were to ask you to describe the structure of a square, what would you say? You would draw a square on a piece of paper and then explain carefully that "squareness" has nothing to do with the color of the paper or the fact that a pencil mark was used, that it does not depend on the length of the sides or the thickness of the line, that if you take away everything from that diagram but a set of spatial relationships you will have a square, a geometric structure. I can do the same thing with a proposition in reply to your question about logical structure. Take all concrete meaning away from the proposition. Suppose it is, "Elephants live longer than birds." Disregard the fact that the proposition deals with elephants and birds. Disregard the fact that it talks about length of life. What have you left? *A type of term related in a certain way to a type of term.* What type of term? What kind of relationship? That is where the logic comes in. There is a structure which transforms a group of words into what we call a "thought," just as there is a structure which transforms a group of lines into a square. Roughly we may say that when the concrete meanings of terms and relations have been removed, all that remains, the structural part, is logical. Just what we mean by this will be clearer as we proceed.

(1) The Logical "Things" of Which a Proposition Is Constructed Are Classes or Individuals

If you care to jot down a list of the terms that appear in the propositions that make up our thinking you will discover that

they fall into two groups. Some of them refer to classes of things or of ideas: elephants, birds, books, presidents of the United States, things made of gold, and so on. In the structure of logic these are symbolized *as classes*, having all of the properties of a class as such, and disregarding completely the things or ideas of which the class is composed. Classes are to logic what numbers are to arithmetic. The number "3" has certain properties which are the same whether you are speaking of three horses or three monkeys or three diamonds or three virtues. Classes have certain logical properties which are independent of their composition.

The other group of terms that appear in propositions are those that refer to individuals: Jumbo, Chanticleer, *Gone with the Wind*, Herbert Hoover, my wedding ring, and so on. In the structure of logic these are symbolized *as individuals*, having all of the properties of individuals as such, and disregarding completely the actual individual to which reference is made. Individuals are to logic what the number "1" is to arithmetic. The number "1" is basic to all of the other numbers; and the individual is basic to classes. Classes are made up of individuals. But when the individual is considered by itself it has certain unique logical properties which are independent of which individual is being considered.

It might seem at first that classes and individuals could be treated alike symbolically, in fact early logicians did not distinguish between them. They would have treated "College towns are quiet" and "College towns are numerous" as exactly alike in structure. But we have already seen that we can get into plenty of trouble if we do not recognize that only an individual town can be quiet and that only a class of towns can be numerous. In most instances the difference between these two types of term can be neglected. The context tells us immediately whether we are dealing with individuals or classes and

there is no ambiguity. If I connect "A" with "white," it is plain that if "A" stands for an individual dress shirt that shirt is white, and that if "A" stands for cakes of soap the individual cakes of soap (a class) are white. "White" can only refer to an individual.

But if I connect "A" with "large" I am open to misunderstanding. Individuals can be large and classes can be large. In the case of the dress shirt there can be no doubt that the individual shirt is large. But if I am referring to cakes of soap I may mean either that each individual cake of soap is large or that there are a great many cakes of soap. In the former case I am treating "cakes of soap" as a group of individuals which have whiteness in common. In the latter I treat "cakes of soap" as an individual class which is a member of the class of large classes. The husband who spoke to his wife of a "beautiful" stage chorus would probably get unsympathetic attention, though he might in all innocence be referring to the brilliance with which the group as a whole executed a dance routine.

A logical class is a term from which all content except that which designates it as a class has been removed. It is a class, but beyond that it cannot be identified. It might be *any* class. And the same is true of the logical individual: it is *any* individual.

(2) The Logical Relations Which Cement the Proposition May Be Classified According to the Number of "Things" Held Together

As regards logical relations the situation is more complex. When you come to examine them in propositions one of the first things you will notice about them is that they relate different numbers of terms. If we count the number of terms

which enter into a given relation we shall have one significant way of classifying the various relations encountered. Most relations, of course, relate only two terms:

A is the father of B
A is taller than B

But there are a distinct number which relate three terms:

A is between B and C
A is the tallest of A, B, and C
A is the son of B and C

Relations which connect more than three terms are rare. Any superlative relation must connect at least three terms and usually connects many more. At a bridge tournament "A *is seated with* B, C, and D" would involve four terms. In basket-ball "is teamed with" brings in five terms. And what about the famous Dionne girls? Can you think of others higher than three? Cases in which only one term appears are special:

A wept
B passes
C walked

For the sake of uniformity these are also spoken of as "relationships," though on the whole they are of less significance for logic than the others.

Or According to the Character Itself of the Relation. Three Fundamental Characteristics: (a) Symmetry

But there is another and in many ways more important aspect of the structure of a logical relation. Suppose we list a number of concrete relationships, each involving two terms:

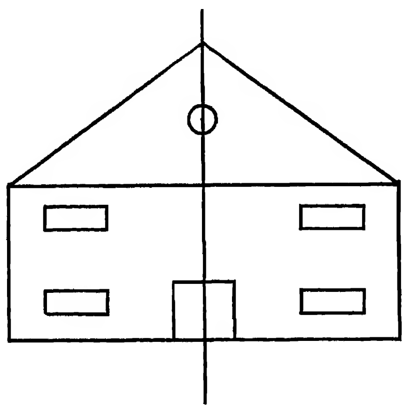
A is the partner of B
A is equal to B

A writes to B
A is east of B
A is heavier than B
A is next to B
A is of the same color as B
A is the spouse of B

Even a hasty examination of these examples will show that structurally they can be divided into two groups:

A is the son of B	A is the partner of B
A writes to B	A is next to B
A is east of B	A is of the same color as B
A is heavier than B	A is the spouse of B

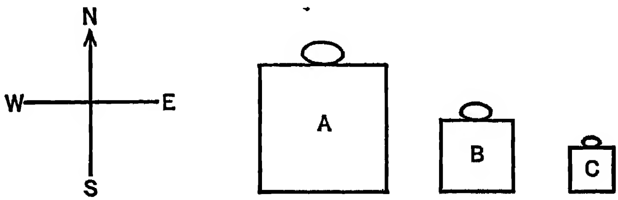
What is the difference between the two? If A is the partner of B, then B is the partner of A. But if A is the son of B, then it is not true that B is the son of A. Some of these relations, the ones in the right-hand column, always work both ways: others, the ones in the left-hand column, do not. In short, the relations in the right-hand column are *symmetrical* relations; and we may designate each as "R_s." A building is symmetrical if it is the same to left and right of a central axis: a figure is symmetrical if it is duplicated either side of a straight line drawn through it:



One side mirrors the other, or the sides coincide if the paper is folded along the axis. And, similarly, some logical relationships are symmetrical in the sense that the terms on either side of the relation R_s (the axis of symmetry) may be interchanged without changing the meaning of the proposition. Whenever we find a proposition that can be symbolized as A R_s B, we know that B R_s A is equally true, no matter what (within these restrictions) the terms and the concrete relation. If a boy is next to a girl, then a girl is next to a boy. If, later, the boy is the spouse of the girl, then the girl is the spouse of the boy. And if Junior's hair is of the same color as his father's, then the father's hair is of the same color as Junior's. But if a boy writes to a girl, it is by no means certain that the girl writes to the boy; and if the boy is east of the girl we can rest assured that the girl is not east of the boy.

(b) Transitivity

If we continue further to examine the left-hand group of relationships we discover two which have properties not possessed by the others in that group. If A is west of B, and B is west of C, then A is west of C; if A is heavier than B, and B is heavier than C, then A is heavier than C:



But if A is the son of B, and B is the son of C, then C is the *grandson* of C. And if John writes to Ruth, and Ruth writes to Henry, there is little chance that John writes to Henry unless

to challenge him to the dueling field. "Is east of" and "is heavier than" are *transitive* relations (R_t): they allow of a logical transition from the first to the third terms in a series. Whenever A R_t B and B R_t C, then we know that A R_t C. Many of our most familiar arguments are, as we shall see, based on the principle of transitivity.

Do you remember the argument about college towns? The major reason why we have to distinguish so clearly between classes and individuals is that the commonest relation among classes, that of class-inclusion, is a transitive relation:

> Small towns are quiet.
> College towns are small.
> ———————————————
> Therefore, college towns are quiet.

while the individual often enters into the relation of class-membership, which is not transitive and hence leads to a fallacy if treated as though it were:

> Small towns are numerous.
> College towns are small.
> ———————————————
> Therefore, college towns are numerous.

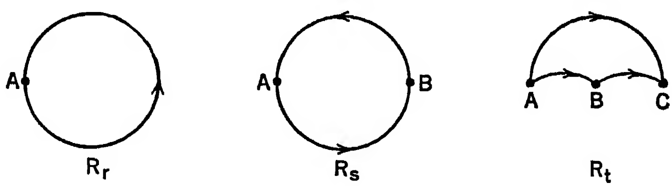
The two relationships may look identical, as in these two cases, but they are logically quite different and must be handled differently.

(c) Reflexivity

The two remaining relationships from the left-hand list may be further distinguished from each other. Neither is transitive: neither is symmetrical. But "writes" is a relationship which might hold between a term and itself: "John writes to himself." Whereas, "is the son of" could not hold between a term and itself: John cannot be his own son. Most relations are irreflexive, but a few have *reflexive* characteristics and are so named. Symbolically they would be expressed as A r_r A. Some

words are generally used reflexively, such as "perjures": others more often used irreflexively, as "defeats." Some are better used reflexively ("pinches"): others better used irreflexively ("praises"). And still other words, like "between" and "is married to" are always irreflexive. In most cases the question as to whether a relation is reflexive or irreflexive can be settled only by examining the context which surrounds it.

R_r, R_s and R_t make interesting company for one another. They are the three most famous logical relationships found in propositions. When taken together they form a triad which has elements of unusual interest. For example, R_r involves only one term: R_s involves two terms, and three terms are necessary for the full expression of the properties of R_t. Furthermore, when illustrated diagrammatically, they form a set of figures which show clearly the differences in these properties:



It is neat too, though sheer accident, that three letters which come together in the correct order in the alphabet can be significantly used to designate them: "r," "s" and "t." It makes you feel the way Robinson Crusoe must have felt when he saw the footprint in the sand.

Various Ways in Which These Three Types of Relationship May Be Combined

These three logical relationships are in one respect like misfortunes. They never come singly. Consider just R_s and R_t for the moment. Some relationships we encounter are both sym-

metrical and transitive: "is the same color as." Some are transitive but not symmetrical: "is heavier than." Some are symmetrical but not transitive: "is the spouse of." And some are neither transitive nor symmetrical: "is the son of." It is profitable and diverting to make a study of a number of concrete relationships familiar to all, to see what their logical characteristics are. Comparatives fall into a general class of transitive, asymmetrical, irreflexive relations: "is heavier than," "is more intricate than." Spatial and temporal relations are generally of this class also. If brotherly and sisterly and "cousin" relationships are excluded, all blood ties provide relations which are intransitive, asymmetrical and irreflexive: "is the grandson of," "is the great-aunt of." Relationships of sameness and identity fall into an entirely different group. Can you identify it?

It is also interesting to work in the other direction, to build a series of purely logical relations and search for the concrete situations which they describe. If we limit ourselves to transitivity and symmetry or their absence, we can have fun plotting out the four possible combinations. What transitive and symmetrical relationships can we find? The members of the House of Lords stand in such a relationship to one another. If Lord Bloomsbury is the peer of Lord Craven, then Lord Craven is the peer of Lord Bloomsbury: and if Lord Craven is the peer of Baron Tweedsbuke, then Lord Bloomsbury is also the peer of Baron Tweedsbuke. The political relationship of these men is of this type, but certainly their financial relationship is not. If the Baron is richer than the other two, they cannot be richer than he is. Hence it is apparent that a particular set of terms can be related one way in one respect, and another in another. Football elevens are related transitively and symmetrically as regards size of team, but not so as regards outcomes of games. We all know that even if Princeton beats Harvard and Yale beats Princeton, Harvard beats Yale.

Consider now transitive but asymmetrical relationships. There are innumerable concrete illustrations. Automobiles as they come off the assembly belt are so related as regards the time of completion. No. 102394 is older than No. 102395, and the latter is older than No. 102396; hence No. 102394 is older than No. 102396. But No. 102395 cannot be older than No. 102394. Cities would have a similar relationship as regards compass direction. Boston is north of New York, and New York is north of Washington; hence Boston is north of Washington. The reader will think of countless other illustrations. In this discussion the words "as regards" appear persistently. They indicate that objects are at the same time differently related in different respects. Those cars coming off the assembly belt would be *symmetrically* related as regards appearance. The cities would be *intransitively* related as regards the dominance of the Republican party. It will be well to bear this in mind.

Relationships which are intransitive and symmetrical are more difficult to find. If my house is within a stone's throw of the highway, and my garage is within a stone's throw of my house, it is not at all certain that my garage is within a stone's throw of the highway, as I discover when I have to shovel out the driveway in winter. But the highway is within a stone's throw of the house, and the house is within a stone's throw of the garage. Things that are within earshot or within sight of one another are similarly related *in these respects*. It is because these are symmetrical but not transitive relationships that the Romans built chains of beacon fires and our Colonials fired chains of muskets to convey news.

Perhaps most difficult of all to find are relationships which are neither transitive nor symmetrical. If a robin eats a worm and a cat eats the robin, the cat may eat the worm; but if a dog bites a man while the man is biting his fingernails, surely the dog is not biting the man's fingernails! Nor is the man biting

the dog (as the newspaperman would wish), or the fingernail biting the man. The latter is surely an intransitive and asymmetrical relationship. Furthermore, if France defeats Germany in war and Germany defeats Russia, it can hardly be said that France defeats Russia any more than it can be said that Germany defeats France. In this last illustration it is interesting to note the question which arises as regards the possible reflexivity of relationships in war. We are coming more and more to see that "defeats" in modern warfare is a relation which turns back on itself. If one nation defeats another it also defeats itself.

We have worked out various possible relationships which are complexes of R_s and R_t. The reader may find amusement in adding R_r (reflexivity) to these, producing new pairs, and then moving on to groups of three, working out the formal systems which they describe. Most of them are subject to graphic representation, if one's imagination is sharp. Try the combinations yourself. There are in all eight different combinations of the three:

Transitive-Symmetrical-Reflexive.
Transitive-Asymmetrical-Reflexive.
Transitive-Asymmetrical-Irreflexive.
Transitive-Symmetrical-Irreflexive.
Intransitive-Symmetrical-Reflexive.
Intransitive-Asymmetrical-Reflexive.
Intransitive-Asymmetrical-Irreflexive.
Intransitive-Symmetrical-Irreflexive.

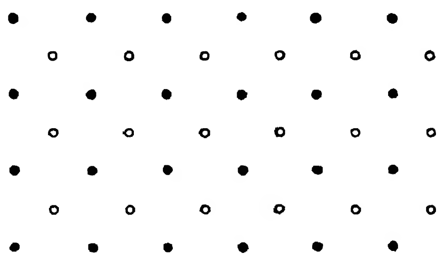
and there is no combination of objects or of ideas which cannot be described according to one of the eight.

These considerations raise the interesting and fundamental question as to whether or not the world exhibits these relationships and hence has been responsible for logic, or the character of human thinking creates the world of these relationships. We live in a rational world. Are we rational because the world is so,

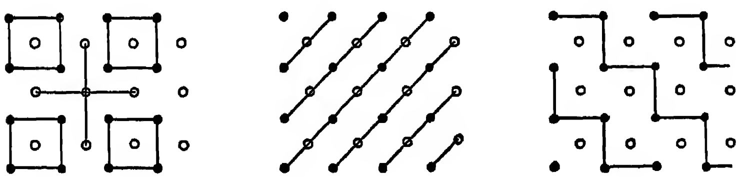
or is the world rational because we are? Men are sharply divided on this question. Its implications are very great, and go beyond the scope of our discussion here to the very heart of philosophy. They have a bearing on the most crucial points in ethics, aesthetics, and metaphysics. Perhaps it is one of those trick questions like that of which came first, the chicken or the egg. Happily we do not have to come to a decision here. In either case, the understanding of formal relationships, and of the systems which exhibit these relationships, is of primary importance in the clear comprehension of the world in which we live. Logic is a game, but also more than a game. It is a fundamental way of understanding.

The Ptolemaic System of Logic

When the following design is placed before you, you will at first see it as alternate rows of black and white dots:



If, however, I draw in certain lines I can make you see squares and crosses, diagonals, or steps:



The possibility of analyzing one and the same structure in more than one manner is characteristic of most systems. Different analyses emphasize different relationships. If properly carried out, all are correct. The same holds true of the system of thought. There are various possible analyses. Some are easier than others to manipulate, some are more adequate than others to the underlying system. Each brings to attention certain characteristics of the structure.

One of the earliest analyses of thought from the logical point of view regarded all terms as *classes*, and reinterpreted all relations as *relations among classes*. It is in many ways to our later analysis of thought as the Ptolemaic system is to the Copernican in astronomy. There is nothing wrong about considering the earth as the center of the solar system, indeed we do it every day in saying that the sun rises in the east, and in building houses so that we can see the sunset from the living-room. But when we consider our solar system in relation to others it is both more adequate and more convenient to take the sun as the center. In the same way, when we are considering the *single proposition* it will be instructive to see what structural characteristics we can find by limiting ourselves to classes and class relationships.

Our terms will be classes. For relationships we shall select the transitive and asymmetrical relation between two terms called "included in"; and the non-transitive, symmetrical relation called "excluded from." These two relations make a combination in itself significant as including the major possibilities in connecting terms:

transitive—asymmetrical
intransitive—symmetrical

The structure that may be found when thought is looked at from this perspective is highly entertaining in itself and, as we

shall see in the following chapter, allows us to bring to light a number of interesting relationships among several propositions. We shall later discover that a broader analysis allows us to work out many more of the relationships among propositions, and may also tell us things about the single proposition otherwise unsuspected, just as the Copernican system tells us more about the universe than the Ptolemaic does. We shall also discover that a broader analysis is easier to manipulate, as is the Copernican system in the field of astronomy. But for the moment let us be Ptolemaic, limit ourselves to classes and class-inclusion and class-exclusion, and see what we find.

The "All" or "Some" of a Class

When I am speaking of a class of objects I may refer to *all* of that class or to *some* of it. For instance, worms and apples. Either all apples are wormy, or some apples are wormy or all apples are not wormy. There is no other possibility. We are not always sufficiently clear in this matter. If I say, "Horses are swift," I leave you in doubt as to whether I am referring to all horses or to some. If I am comparing them to human sprinters I mean "all": if I have just lost a daily-double I probably mean "some." To convey our meaning accurately we must be explicit. When we are paying attention to logical form in this analysis we must *always employ one of the two words,* "all" or "some."

But what to do when I want to speak of Old Dobbin? She may be *some* horse, but she is not either "some horses" or "all horses." What shall we do when we want to speak of an individual in a Ptolemaic system which uses only classes as terms? It takes a little juggling. But the fact of the matter is that in more senses than one Old Dobbin is in a class by herself! Logically and Ptolemaically she is a class of one member.

Within that class she is the whole cheese, in fact she *is* the class, *all* of the class. The "all" is implicit. When we say that the "old gray mare ain't what she used to be," we mean the whole mare and not just a creaky knee joint or a few broken teeth. "Caesar is ambitious" is actually and more exactly, "(All) Caesar is included in ambitious men." "Xantippe was no beauty" becomes an example of class exclusion: "(All) Xantippe is excluded from the class of beautiful women." Katisha may have had a beautiful elbow but taken as a class of one member she caused Ko-Ko to tremble. So remember that when you speak of an individual you are referring to a special type of class and using the word "all" by implication.

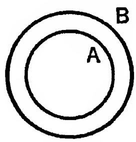
The "Included In" and "Excluded From" of Related Classes

When I am speaking of the relations between classes *as classes* I may speak of *inclusion* or of its opposite, *exclusion.* Again, though this is more difficult to understand, there is no other possibility. Consider for a moment two classes: (1) the class of cats, and (2) the class of animals that purr. Cats (some or all) are either included in the class of animals that purr or they are excluded from it. If you say that some cats purr and some do not, you are actually asserting two separate propositions: "Some cats are included in the class of animals that purr" and "Some cats are excluded from the class of animals that purr." If you say that the class of cats is identical with (i.e. has the same members as) the class of animals that purr, you are also asserting two separate propositions: "All cats are included in the class of animals that purr" and "All animals that purr are included in the class of cats." Any relationship between classes *as such,* no matter how complex, can be reduced to these two, inclusion and exclusion.

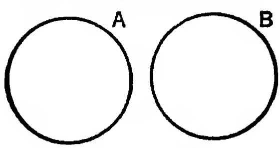
The "All" or "Some," and the "Included In" or "Excluded From," Combined in the Single Proposition. Four Basic Propositions

Now let us suppose that we are dealing with two classes of objects, any two classes; horses and cows, men and women, ships and sealing-wax, cabbages and kings. Designate one class as the "A" class (it makes no difference which) and the other as the "B" class. Taking A first (again, it makes no difference which, for the form will always be the same), we may speak either of *all of A* or of *some of A*, and we may relate it to B either by inclusion or by exclusion. There are four possible combinations of these pairs, *and only four:*

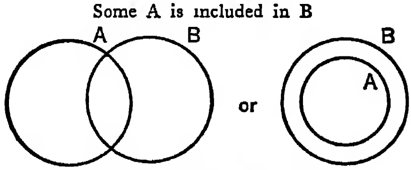
All A is included in B



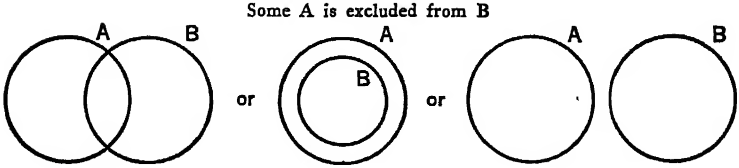
All A is excluded from B



Some A is included in B



Some A is excluded from B



Or, to go at the matter in a different manner, using only the all-some variable, we may relate *all of A* to either *all of B* or *some of B*, or *some of A* to either *all of B* or *some of B:*

all of A	to	some of B
all of A	to	all of B
some of A	to	some of B
some of A	to	all of B

Already we are getting a little design: all-all and some-some on the left, and some-all and some-all on the right. Again, no other combination is possible. And it is not difficult to see that the two groups of four possibilities express the same thing in different ways:

If I say that all men are included in rational animals, I need to know about *all* men but only about *some* rational animals (those that are men).

If I say that all men are excluded from rational animals, I need to know about *all* men and also about *all* rational animals (to be sure that none of them are men).

If I say that some men are included in rational animals, I need only to know about *some* men and about *some* rational animals, without studying either class exhaustively.

If I say that some men are excluded from rational animals, I need only know the "*some* men" in question, but I must know about *all* rational animals to be sure that the men in question are excluded from them.

Which is another way of saying, of course, that if you include one class of objects in another you include it as only a part (or *some*) of that class; while if you exclude one class from another you exclude it from *all* of that class. If you *include* Fords (all or some, it makes no difference) among automobiles, you make them *part* of a larger class. But if you *exclude* Austins (again, all or some) from automobiles you set them apart from the *entire* class of automobiles.

The most interesting thing to note at the moment is that all propositions about classes can be expressed in one, or a combination of, these forms. Consider a few examples. The most famous one, traditionally a part of every logic book, is this: "He jests at scars who never felt a wound." We see immediately that this is an inclusion. What does it mean?

All who jest at scars are included in those who never felt a wound? Obviously this is not the meaning intended, for in this form the statement does not exclude the possibility of men who had never been wounded yet did not jest at scars.

Some who jest at scars are included in those who never felt a wound? No, for this form does not exclude that possibility either.

Some who never felt a wound are included in those who jest at scars? No, and again for the same reason.

All who never felt a wound are included in those who jest at scars? Yes. This carries all of the meaning intended without saying too much.

Of course this is a matter of interpretation. What *was* meant originally? I have heard endless arguments about this. Some claim that the classes were intended to be made identical: All who jest at scars are included in those who never felt a wound *and* all who never felt a wound are included in those who jest at scars. This only goes to show that in its original form the sentence is ambiguous. It is like "All that glisters is not gold" in this respect. The statement becomes clear and unequivocal only when put in strict logical form.

One must be particularly careful when the word "only" appears. If the sign at the entrance to a swimming pool says "Only men admitted on Tuesday," it obviously does not mean that *all* men are admitted on that day. It means that all who are admitted on Tuesday are men. The classes have to be reversed before the meaning is clear. There are many other

examples. Among the best is the familiar, "All's well that ends well." One is tempted to make this into, "All things that are well are included in things that end well," whereas the true meaning is just the reverse. "All things that end well are included in things that are well."

Meet A, E, I, and O, and Be Careful How You Treat Them at First

The four basic propositions which we have just described have been designated in the traditional logic by giving them the first four vowels in the alphabet as names. Thus:

An *A* proposition:	All A is included in B.	All $A < B$
An *E* proposition:	All A is excluded from B.	All $A \nless B$
An *I* proposition:	Some A is included in B.	Some $A < B$
An *O* proposition:	Some A is excluded from B.	Some $A \nless B$

It is important at the beginning of a study of the internal structure of a single proposition to obey two rules in restating propositions which are not originally in strict and clear form. Few are. (1) Always use "all" or "some" and no other word at the beginning of the proposition, unless you refer to an *individual*, in which case, as we have seen, "all" is implicit. (2) Always use either "included in" or "excluded from," or the symbols for these relations. And be certain that the two classes under discussion *can* be so related. It may seem preposterous, but beginners have analyzed "All cats have tails" into "All cats < tails." Of course it should be, "All cats are included in *things which have* tails." It is helpful to remember that the more often such phrases as "things which are" and "men who have" appear in your restatements of propositions the more confidence you may have in the accuracy of your analysis.

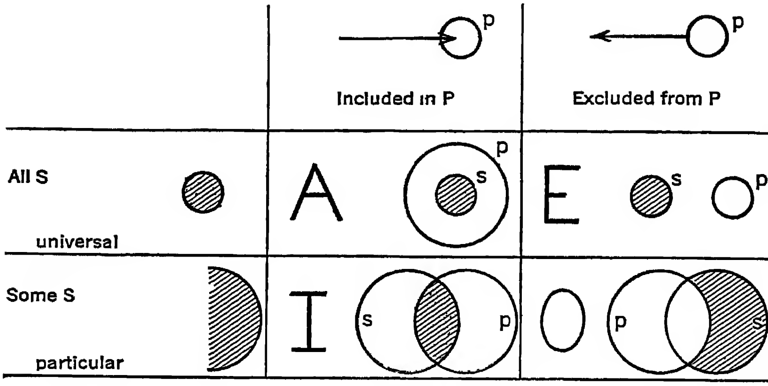
This business of stating propositions accurately is something like walking a tight-rope. The beginner should watch his step with painstaking care, but the experienced artist is accustomed to the thing and may allow himself greater freedom. The seasoned circus performer may turn somersaults on his rope with impunity, but his skill is the result of long hours of practice. You will need practice, too, in handling propositions. When you become skillful you will no longer have to observe the elementary cautions.

A Pattern Begins to Emerge

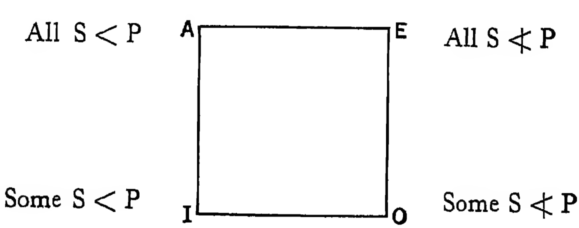
Now the fun commences. We have in our possession four propositions, each with a unique personality, yet all much alike in the elements of which they are composed. It is as if we had struck the notes C-E-G-C on a piano, each a distinct tone, yet all alike in being musical notes. We strike the four notes together and discover to our pleasure that they make a harmony. Bring the four propositions together into one figure and you will be amazed at their harmony of interrelation. When taken together A, E, I, and O form a pattern which is so neat that one suspects a God of Logic of having arranged all of the details ahead of time. If you are interested in patterns you will want to add this one to your collection. It is one of the best.

The A-E-I-O pattern is a geometric one. You will see why this is the case if you will reverse the process we have just completed and analyze the four propositions into their elements again for a moment. Noticing that two of the propositions begin with "all" (and hence are called *universal* propositions) and that two begin with "some" (*particular* propositions); and that two employ inclusion, and two, exclusion; we may compose an interesting diagram in which the all-some variation is

indicated vertically and the inclusion-exclusion variation, horizontally. Thus:



Just as the number three, in marriage relationships and elsewhere, suggests a triangle; so the number four puts us in mind of a square. We have four propositions. Our diagram shows clearly that we are dealing with things which have two dimensions (the *all-some* and the *inclusion-exclusion*) and four parts. How about arranging A, E, I, and O at the four corners of a square, the universal propositions being in the top corners, the particular propositions at the bottom, inclusions to the left, and exclusions to the right?



In traditional logic this has been called the Square of Opposition. Its properties are really amazing.

A, E, I, and O Are Very Much on the Square

A square is a symmetrical geometrical figure and, similarly, the relationships between A, E, I, and O propositions are symmetrical. For example, given the same subjects and predicates, A has the same relationship to I that E has to O, and *vise versa:*

If A is *true*, then I is *true*.[1]	If all apples are red, then some are red.
If E is *true*, then O is *true*.	If no apples are red, then some apples are not red.
If I is *true*, then A is *doubtful*.[1]	If some apples are red, we do not know whether or not all apples are red.
If O is *true*, then E is *doubtful*.	If some apples are not red, we do not know whether or not none are red.

One way of expressing this symmetry would be to say that I is like A except that it is a weaker statement, using "some" instead of "all." For the same reason O may be considered a weaker version of E. It is not difficult to work out the other four relationships between vertical pairs:

If A is *false*, I is *doubtful*.
If E is *false*, O is *doubtful*.

If I is *false*, A is *false*.
If O is *false*, E is *false*.

Note the double symmetry: (1) that between the pairs in each group of four, and (2) that between the two groups of four. There are so many similarities of relationship to be seen that one does not appreciate them all immediately. The most important ones can be diagrammed:

[1] See p. 115-6 for an important qualification with regard to this property of the Square.

A		*I*	*E*		*O*
True	⟶	True	True	⟶	True
False	⟵	False	False	⟵	False

The system into which these relationships fall provides one of the most marvelous of all of the pages in logic.

The symmetry between horizontal pairs is equally interesting and probably more significant. It may be expressed in the following manner:

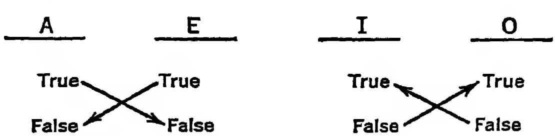
If A is *true*, E is *false*.	If all apples are red, it is false that no apples are red.
If E is *true*, A is *false*.	If no apples are red, it is false that all apples are red.
If I is *false*, O is *true*.	If it is false that some apples are red, then it is true that some are not red.
If O is *false*, I is *true*.	If it is false that some apples are not red, then it is true that some are red.

Another and perhaps more compact way of saying this is to say that *A and E cannot both be true at the same time*, and that *I and O cannot both be false at the same time*. The other four relationships of these horizontal pairs have the same symmetrical properties:

If A is *false*, E is *doubtful*.
If E is *false*, A is *doubtful*.

If I is *true*, O is *doubtful*.
If O is *true*, I is *doubtful*.

A and E may both be *false* at the same time, or one false and the other true. I and O may both be *true* at the same time, or one false and the other true. The symmetry here is a sort of reverse symmetry, as the diagram will indicate:



The more one studies these things the more one perceives.

There is one more quality of symmetry in these four propositions, one that is expressed along the two diagonal lines of the square. This is a tighter relationship in the sense that the doubtful does not enter:

If A is *true*, O is *false*. If all apples are red, it is false that some are not.

If E is *true*, I is *false*. If no apples are red, it is false that some are red.

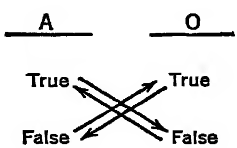
If I is *true*, E is *false*. If some apples are red, it is false that none are red.

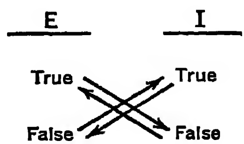
If O is *true*, A is *false*. If some apples are not red, it is false that all are red.

And there is the same diagonal relationship between falsity and truth as between truth and falsity:

If A is *false*, O is *true*.
If E is *false*, I is *true*.
If I is *false*, E is *true*.
If O is *false*, A is *true*.

And again the salient features may be expressed in diagrams, in these cases there being four arrows instead of two:





In short, we may say that if any one of the four propositions is true we know with certainty that its diagonal is false, and that if any one of the four is false, its diagonal is true. Nothing could be neater.

Contrary and Contradictory in Children and in Propositions

Most of us make the careless mistake of using the words "contrary" and "contradictory" interchangeably. Of a pair of propositions which run counter to one another we are apt to say either that they are contrary or that they contradict each other. In doing so we are missing an important distinction. Consider the following pairs of statements:

All farmers are happy.
No farmers are happy.

All farmers live to a ripe old age.
Some farmers do not live to a ripe old age.

That within each pair the propositions oppose each other is perfectly clear. In either case, if one of the propositions is true the other must necessarily be false. But the two pairs represent different types of opposition. In the former case *both* may be false: in the latter case only one may be false, and the other will always be true. The former is an example of *contraries*, and holds between A and E propositions; the latter is a case of *contradiction*, and holds between A and O, or between E and I propositions.

The difference may be illustrated in terms of the insubordinate child. The contrary child is the one who wants to go outdoors when it is indoors, who is perfectly happy until Johnny rides his tricycle, who wants peas when served beans. We all know him. But the contradictory child is the more serious, and if possible the more exasperating, case. He is the one who stamps his foot and says, "No! I don't want to!" or "I won't!" The contrary child is unhappy because he cannot do two things at once. Contrary propositions are unhappily related because they cannot both be true. But they may both be false, just as the

child may not eat either peas or beans. The contradictory child simply refuses to do what he is told. So contradictory propositions simply deny one another's truth.

The distinction may, of course, be carried out to apply to propositions whose relations depend on content:

She is a blonde.
She is a brunette.

He is in the house.
He is outdoors.

In these instances the relationships are contrary and contradictory, respectively, because of the facts of the case. Maybe she is a red-head! If he is not indoors he is certainly outdoors, provided you define the two accurately and precisely. There is an interesting illustration here of the difference between *logical* and *material* relationships, and ground for endless speculation. For example, "contrary" seems clearly to be either material or logical, but there is some reason to believe that "contradictory" must always be ultimately logical in origin. These problems we leave to the reader.

One further distinction needs to be considered. There are two kinds of contraries, as the following will illustrate:

All wombats have ringed tails.
No wombats have ringed tails.

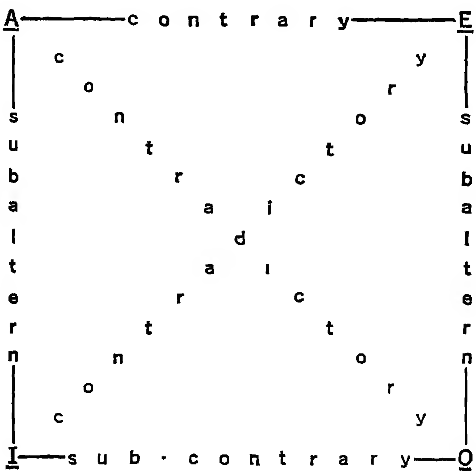
Some men smoke.
Some men do not smoke.

The former is a violent opposition: the latter is relatively mild. Martin Luther went around belligerently saying, "All good Christians are Protestants," ready to smite any one who dared assert that no good Christians are Protestants. How different the history of Europe might have been if Luther had been content with "Some good Christians are Protestants" and the Pope

with "Some good Christians are not Protestants." In either case there is disagreement, but the difference is notable, as Erasmus pointed out. If you say to your opponent, "We may both be wrong but we cannot both be right," you probably do so with fire in your eye. If you say, however, "Perhaps we are both right; and at least we cannot both be wrong," you will do so in a spirit of comradeship. In the traditional logic the milder type of contrary is called a "*sub-contrary.*"

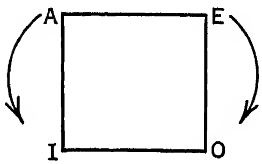
The Completed Square of Opposition

We are now in a position to construct a more comprehensive diagram of the Square of Opposition, naming the various types of relationship which are exhibited. Understanding its full meaning is like riding a bicycle: once you get it you never forget:

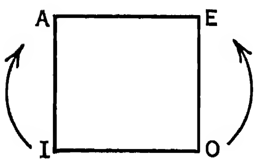


There is one characteristic of this extraordinary square yet to be mentioned. It is implicit in what has already been said, but

is worth mentioning separately if only for the opportunity it gives us to marvel at the eloquence of the device. It is this. Start anywhere on the square and go either clock-wise or counter-clock-wise. You will find exactly the same relationships to hold between the proposition with which you started and the other three, *and in the same order*, as if you had started on the other side of the vertical axis and moved in the opposite direction. This sounds complicated, but an illustration will show this curious property more clearly·



If A is true	*If E is true*
I is true	O is true
O is false	I is false
E is false	A is false



If I is false	*If O is false*
A is false	E is false
E is true	A is true
O is true	I is true

The various other combinations we shall leave to the reader. There are six others. See if you can work them out.

Putting Propositions in Reverse. E and I Are Docile

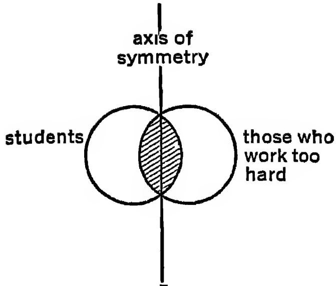
Sometimes people are interested to know in what ways propositions can be converted so that the predicate appears as subject and the subject as predicate. This conversion will be particularly useful when we study the syllogism in the next chapter. If we know that all apples are red, what can we say about red

things in relation to the class of apples? In order to find this out we must recall our discussion of the all-some properties of propositions. We said that these properties were divided among the four propositions in four pairs:

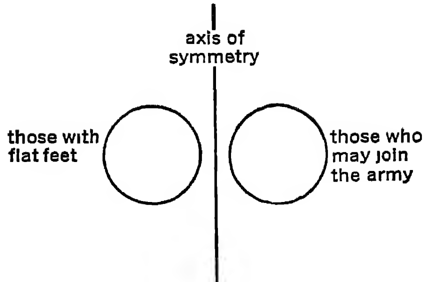
A.	*All* A is included in *some* B	(all-some)
E.	*All* A is excluded from *all* B	(all-all)
I.	*Some* A is included in *some* B	(some-some)
O.	*Some* A is excluded from *all* B	(some-all)

Some things can be reversed without changing their properties. If we reverse the board of a seesaw it is still a seesaw. A cigarette is reversible—unless it has a cork tip. But you cannot reverse a sailboat or a pipe. The reversible things are, in general, symmetrical. And, in a rough sense, the same principle holds as regards propositions. Those which are most easily converted will be the symmetrical ones.

A glance at the list above shows that of the four, two are symmetrical as regards "some" and "all." The I proposition has "some" in both subject and predicate; and the E proposition has "all" in both subject and predicate. To use a more technical terminology, in an I proposition both terms are undistributed; and in an E proposition both terms are distributed. This being the case these two can be reversed, or converted, without altering their meanings. If I can assert as true the statement that some students work too hard, I am in a position to assert as true also the statement that some who work too hard are students. Also, if the regulations tell us that all who have flat feet are excluded from those who may join the army, then by implication they also tell us that all who may join the army are excluded from those with flat feet. This characteristic of I and E propositions is made doubly clear by the symmetry of the diagrams which can be drawn to illustrate them. Using the examples cited above:



I



E

In other words, in stating I and E propositions it makes no difference which term comes first. They are the same either way. The process of conversion is so easy that it is called conversion *simpliciter*.

But A and O Give Trouble

The situation is quite different with A and O propositions and must be handled differently. They are not symmetrical as regards the distribution of their terms: in each "all" appears on one side and "some" on the other. It is easy to assert that all who vote the Republican ticket deserve consideration for political jobs; but it is quite wrong to infer from this that all who deserve consideration for political jobs are those who voted the Republican ticket. If all cats are included in the class of animals which have claws, it certainly does not follow that all animals which have claws are included in the class of cats. This is an altogether stupid, though not infrequent, line of argument. All Reds are enemies of the country, therefore all enemies of the country are Reds! Emotion, not reason, is responsible for such utterances.

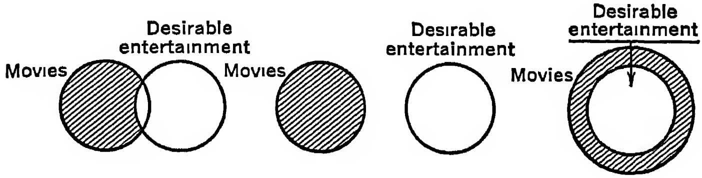
In dealing with an A proposition, if it is converted only part of the meaning may be retained and the result must take the form of an I proposition. If it is true that all cats have claws,

we may certainly assert with confidence that *some* of those animals which have claws are included in the class of cats. If all Reds are enemies of the country, it is also true that some who are enemies of the country are Reds. In short, when an A proposition is converted it becomes an I proposition, and is of lesser importance as an assertion. The I proposition says less. And we know that it says less because we cannot get back from it to the original A.

If, for example, we should start out with "Some animals with claws are cats" we could never get to "All cats have claws" without additional information. Less knowledge is at our disposal. You might assert the I proposition after having examined only your cat and the cat next door; but you could not assert the A proposition without having examined all cats. And, similarly, if you know that some who are enemies of the country are Reds, you are far from being able to assert that all Reds are enemies of the country. The individuals whom you studied as a basis for the former assertion might be enemies not because they are Reds but because they have been mistreated here. There is a world of difference between the two assertions, as many district attorneys might learn to the advantage of their clients if not of themselves. For a reason too remote to concern us here, the conversion of an A into an I proposition is called conversion *per accidens.*

When we come to tackle the O proposition we suffer embarrassment. It just happens that if you assert that some of one class is excluded from all of another, there is nothing that can be said about the second class in its relation to the first. Consider a concrete case. If you discover that some moving pictures are excluded from the class of desirable entertainments, there is nothing you can say with certainty about desirable entertainments which will link them with the class of moving pictures. It may be true that all desirable entertainments are movies, that

some desirable entertainments are movies, or that no desirable entertainments are movies. You just cannot tell. This is best made clear by examining three diagrams, *any* one of which illustrates the original O proposition:



1 2 3

Without knowing which of these three situations is the case, it is impossible to make with assurance any statement which will have "desirable entertainment" as the subject and "moving pictures" in the predicate, for in each of the three the relation of the former class to the latter is different.[2]

Summary

This completes our analysis of the single proposition on the basis of *classes*, and their relationships to each other of *inclusion* and *exclusion*. We have seen that there are four possible combinations of these terms and relations, giving four basic propositions, that these four propositions can be placed at the four corners of a square of opposition, and that these four

[2] The problem of *conversion* is but one aspect of a larger problem called that of *immediate inference*, the study of what other propositions can be inferred from a single proposition. Another aspect is that of *obversion*, in which the "included in" is changed to "excluded from," the subjects and predicates remaining in the same position. For example.

All A is included in B is equivalent to *All A is excluded from not-B*
Some A is excluded from B is equivalent to *Some A is included in not-B*

Then there are obverted converses, partial contrapositives, full contrapositives, partial inverses, and full inverses. The reader who is interested in knowing more about *immediate inferences* will find a table of them in the appendix.

corners boast an orderly set of logical interrelations. This system is interesting in itself. We hope that the reader has found it so. But we have only begun to show the possibilities of class analysis. When we put propositions together to form arguments another whole system emerges, the system of syllogistic reasoning. This second system when understood is even more fascinating, and to it we shall devote the next two chapters.

Chapter Two

THE SYLLOGISM: A PTOLEMAIC VIEW OF CLASSES

THE most rational man is a coward about his dentist appointments. But if you can get him to agree that if he will see his dentist twice a year he will minimize both the cost and the anguish of his visits, then you can persuade him that he *should* make an appointment immediately, for he will hardly deny his interest in reducing both financial and physical pain. You have helped him reach a conclusion by forcing him to recognize and accept two premises. Dentists assure us of the truth of the first premise, and human nature assures us of the truth of the second. We might put the argument more formally thus:

All who wish to reduce the pain of dentistry should see the dentist at least twice a year.
He wishes to reduce the pain of dentistry.
___
Therefore, he should see the dentist at least twice a year.

This type of argument is known as a *syllogism*. It is made possible by the fact that certain terms and relationships are carried over from the premises above the line to the conclusion below. A complete analysis of the argument we have cited would bring it into the following form:

(All who wish to reduce the pain of dentistry) $<$ (those who should see the dentist at least twice a year)
(He) $<$ (all who wish to reduce the pain of dentistry)

(He) $<$ (those who should see the dentist at least twice a year)

One term or class, "all who wish to reduce the pain of dentistry," appears in both premises. The other two, "he" and "those who should see the dentist at least twice a year," appear both in premises and conclusion. To reduce the argument to symbols:

$$\begin{array}{rc} & A < B \\ & \underline{C < A} \\ \therefore & C < B \end{array}$$

The class A is included in the class B, and the class C is included in the class A. Therefore, the class C is included in the class B. It is not difficult to see that this will hold regardless of what A and B and C stand for—anything from aard-varks to zymologists will do.

What Is a Syllogism?

The syllogism may be defined as an argument containing two premises and a conclusion, in which three and only three classes appear. One of the classes will always be common to the premises, and the other two will appear both in a premise and in the conclusion. And the relationship between the classes is always either inclusion or exclusion.

Fortunately for the aesthetics of language, but unfortunately from the point of view of clear thinking, syllogistic arguments seldom appear in exact and correct form. A precise syllogism is like a tuxedo—formal, stiff and uncomfortable. Speech and literature must appeal. We seek conviction and flow of language: the man who speaks entirely in full-blown syllogisms will lose his audience before he has convinced it. Hence it is

the custom (outside of textbooks) to express an argument in such a manner that other and more humane virtues come to the fore. The technically correct syllogism is as unnatural as a lion in a cage. The syllogism in its native haunts is a different animal. Human nature *vs.* logic, again!

If I were trying to convince an audience that there are times when all men can be fooled I should probably say something like this: "Look here. You will agree with me that any one who is subject to passions can be fooled at times; and hence you cannot deny that sometimes we are all fooled." Logically my argument would be this:

(All who are subject to passions) < (those who can be fooled)
(We) < (all who are subject to passions)
∴ (We) < (those who can be fooled)

But who would care to hear it stated just that way? Much more effective to soften the wording and suppress the obvious premise. Sometimes we suppress both the obvious premise and the conclusion. In trying to show that some men are happy, I might simply point out that all who live in the country are happy. If my neighbor is trying to get me to vote the Republican ticket, all he bothers to tell me is that the Republican ticket has the best candidate. We both assume implicitly the importance of voting for the best candidate, and I am perfectly aware of the conclusion toward which I am being led. My neighbor would be a bore if he were too explicit.

Like the lion, the syllogism in its native haunts is dangerous. It is dangerous for two reasons. Maybe the suppressed premise is not one to which we will all immediately subscribe. "Of course you must go to the dance; if you do not, you will not be popular!" Is going to dances the only way of being popular? And what is the importance of being popular? "I must have a

new hat: Mrs. Smith next door just bought one." Well, what of it?

Or it may be that when the suppressed premise is brought out into the open and placed beside the one explicitly stated, the structure of the argument is shown to be faulty. "Look at your hair, you bad boy! You have been in swimming!" How often the switch is more powerful than logic. "He isn't crazy: he would not teach if he could do anything else." Examples can be enumerated without end. To change our metaphor, you have to know a syllogism when you see it coming down the street all dressed up in its Sunday best. If it is a bad one the cloven hoof may be well covered by shoe leather; and if it is good you must be wary of judging it so simply because of an engaging smile, an earnest step, or a threatening club.

In dealing with syllogistic arguments here we shall try to avoid the caged-animal appearance. We shall do this not by suppressing premises, for that would make our problem too difficult. But we shall try to use language as it appears in everyday employment, not the way the logician might like to have it used.

Sixty-four Syllogisms Standing on a Shelf

Any mathematician will tell you that if you draw three cards from an ordinary bridge deck there are sixty-four different combinations of suits which you can get if you take the order of the drawing of the cards into account. Let us choose another figure. If there were four mouse holes and three mice were scampering to them in hasty flight from Tabby, there would be sixty-four different ways in which they might find refuge. Figure them out!

What has this to do with the syllogism? We have seen that a syllogism is an argument composed of *three* propositions (the

mice), two of which are premises and the third a conclusion. We have also seen by our analysis of the internal structure of a proposition that there are *four* possible types of proposition, and only four (A, E, I, and O, the mouse holes), which can enter into the construction of a syllogism. We have four propositions, and we want to know how many different combinations of three we can get from them. It is a simple mathematical problem, like the ones above. And the answer is the same, sixty-four.

Now for a surprise. Take any one of the sixty-four combinations at random; a syllogism made up of the last three types of proposition, E-I-O, let us say:

E No dogs are animals with antlers.
I Some dogs are animals with hind legs.

O Therefore, some animals with antlers do not have hind legs.

The absurdity of such an argument is apparent; it is not even remotely plausible. But, curiously enough, if we take the same three propositions in the same order, E-I-O, and rearrange the internal elements, we get an argument that is perfectly valid!

E No animals with antlers are dogs.
I Some animals with hind legs are dogs.

O Therefore, some animals with hind legs do not have antlers.

There is something odd going on here. What are we to think?

It is as if we had suddenly discovered that we are drawing cards from several packs, or that our mice in escaping from Tabby could use more than one technique in getting to the nearest exit. There are many more than sixty-four possibilities in cards, escapes, or combinations of propositions. Obviously, if E-I-O can give one argument one time and another argument another, there are further refinements to take into account. The internal properties of the propositions make a difference, too.

When We Come to Terms with the Syllogism, Sixty-four Is Only a Beginning

Suppose we examine those two arguments about dogs and antlers and hind legs more carefully. Something has escaped us. We remember that every syllogistic argument contains three terms, two of which appear once in the premises and once in the conclusion, and the third in each premise. It will help matters to give these terms names. The term which is restricted to the premises (always above the line) is called the *middle term* (M), and the other two are called the *subject* (S) and *predicate* (P) *terms*, depending on whether they appear in the subject or the predicate of the conclusion.

If we will look at the terms in these two arguments and give them their proper names, we shall see immediately what has happened in moving from one to the other. The first argument looks like this:

All *M* is excluded from *S*.	M	S
Some *M* is included in *P*.	M	P
Therefore, some *S* is excluded from *P*.	S	P

While the second is of the form:

All *P* is excluded from *M*.	P	M
Some *S* is included in *M*.	S	M
Therefore, some *S* is excluded from *P*.	S	P

The difference between the two is a difference in the position of the terms. It is also the difference between a valid argument and an invalid one!

How many packs of cards have we? In how many ways can the mice get to the holes? In order to examine all of the possibilities in syllogistic argument, we shall want to know how many different combinations there are as regards the positions of the terms. Fortunately there are only a few. The subject

and predicate terms, by their very definition, will always appear in the same place in the conclusion, the subject first and the predicate second. This simplifies things: we have only the positions of the terms in the premises to consider. And to this the place of the middle term is the key.

If the middle term comes first in the initial premise it may come either first or second in the other premise: and if it comes last in the initial premise it may come either first or second in the other premise. Four possibilities, and four combinations. There are no others:

$$\begin{array}{cc} \mathrm{M} & \mathrm{P} \\ \mathrm{S} & \mathrm{M} \\ \hline \mathrm{S} & \mathrm{P} \end{array} \qquad \begin{array}{cc} \mathrm{P} & \mathrm{M} \\ \mathrm{S} & \mathrm{M} \\ \hline \mathrm{S} & \mathrm{P} \end{array} \qquad \begin{array}{cc} \mathrm{M} & \mathrm{P} \\ \mathrm{M} & \mathrm{S} \\ \hline \mathrm{S} & \mathrm{P} \end{array} \qquad \begin{array}{cc} \mathrm{P} & \mathrm{M} \\ \mathrm{M} & \mathrm{S} \\ \hline \mathrm{S} & \mathrm{P} \end{array}$$

Some one who is alert, but not alert enough, will ask right away why we cannot get four more combinations by reversing the order of the premises, as for example:

$$\begin{array}{cc} \mathrm{S} & \mathrm{M} \\ \mathrm{M} & \mathrm{P} \\ \hline \mathrm{S} & \mathrm{P} \end{array}$$

This fifth combination seems to be a sort of hybrid between the first and the last in our series of four. Is it unique or a duplicate?

Does it duplicate the fourth? Its middle terms are arranged like the middle terms in the fourth of the series. But it is not identical with the fourth, for the S and the P are interchanged. Where S appears in the fourth, P appears in this fifth; and where P appears in the fourth, S takes its place in the fifth. In order to make the fifth look like the fourth the S and P would have to be reversed in the latter, and its conclusion would become *P is S*. This, of course, would alter the entire argument. Surely this hybrid fifth arrangement of terms *is* different from the fourth.

Does it duplicate the first? Here lies the catch. Actually this fifth suggestion is exactly like the first as an argument, except that the premises are differently ordered. In both cases the premises are *M is P* and *S is M*. Does this difference in the order of the premises alter the argument? A little reflection will show what the first and the fifth arrangements of terms offer identically the same argument, the same structure.

We have uncovered an important principle in the understanding of the syllogism. The principle is this, that *it makes not a whit of difference to the structure and validity of a syllogism which premise is stated first.* Since no chapter on the syllogism would be complete without mention of the old chestnut, let us bring it in now by way of illustration. The two following arguments are exactly the same:

All men are mortal.	Socrates is a man.
Socrates is a man.	All men are mortal.
Therefore, Socrates is mortal.	Therefore, Socrates is mortal.

A couple of circles on paper, the old test, will show this beyond the possibility of a doubt. You are basing your argument on two premises. Psychologically it may be important, but logically it is quite unimportant which you state first. Each has the same meaning whether it is stated first or second. Hence the hybrid fifth is really the first arrangement in disguise.

In order to simplify the system, therefore, traditional logicians have established the convention that in stating an argument *the premise which contains the predicate of the conclusion,* called the *major* premise, *shall always be stated first.* Hence we may assert the form in which these arguments shall be expressed to be the following:

Major premise	(containing predicate of conclusion)
Minor premise	(containing subject of conclusion)
Conclusion	

The convenience is obvious, reducing the number of combinations involving the position of the middle term from a possible eight to the four mentioned first. These four combinations are called the *figures* of the syllogism, and they have been given numbers:

I		II		III		IV	
M	P	P	M	M	P	P	M
S	M	S	M	M	S	M	S
S	P	S	P	S	P	S	P

The reader will have to work out his own mnemonic device for fixing these in mind. I try to remember that M appears only on one side in II and III, that in the *first* figure M appears *first* in the *first* premise, and that in the *second* figure M appears *second* in the *second* premise. Have you a better suggestion?

Two Hundred and Fifty-six Syllogisms Standing on a Shelf. And All But Twenty-four Are Left There

So we are left with only four different ways in which the terms may be arranged in any one trio of propositions. But four is four. It is as if we had discovered that in drawing groups of three cards we found that we were drawing from four different packs, and that we had to watch both the suit of the cards *and* the pack from which they came. Or it is as if we had discovered that the three mice not only had four holes into which to escape but could get to the holes by walking, running, flying or roller-skating. The problem is four times more complex!

At this point we have on our hands a depressing number of possible syllogistic arguments. There are sixty-four combinations of A, E, I, and O propositions; and each can appear in each of four figures. Two hundred and fifty-six different combina-

tions in all! Fortunately for us, all but twenty-four of these are grossly invalid. The student who wishes to work out all two hundred and fifty-six, drawing circles for each, can find which the twenty-four are. We shall not be so unkind as to let him do it. It has all been done for us. Why do it again? The twenty-four which deserve attention are these:

I	II	III	IV
A A A	E A E	A A I	A A I
A I I	A E E	A I I	I A I
E A E	E I O	I A I	A E E
E I O	A O O	E I O	E A O
A A I	E A O	O A O	E I O
E A O	A E O	E A O	A E O

They are nicely distributed, six in each figure. How much system can you find here? Two of the combinations, EIO and EAO, appear in each of the four figures. Some, like AAA, appear only once. Anything more?

We have done well to reduce the scope of our attention from two hundred and fifty-six to twenty-four. It is as if we had sought the few combinations of three cards that were straight flushes, or the few quickest ways for the mice to get to their holes. These are the few valid syllogistic arguments. Perhaps we should be glad that the number is so small.

Twenty-four Syllogisms, and Five Are Returned to the Shelf

As a matter of fact the number of syllogisms that will interest us is even smaller than twenty-four. One further simplification is possible. Among these twenty-four arguments there are several which tread so heavily on the toes of others that they must be banished from the list. In short, they are redundant.

Limiting ourselves for the moment to the first figure, consider AAA and AAI:

A	All cats are felines.	A	All cats are felines.
A	All tabbies are cats.	A	All tabbies are cats.
A	Therefore, all tabbies are felines.	I	Therefore, some tabbies are felines.

The second (AAI) is more or less a special case which might without loss be included under the first (AAA). Note that the premises are exactly the same, and that they allow us to draw the strong conclusion that all tabbies are felines. We remember from what we learned in Chapter One about the relation between A and I propositions, that the I proposition is just a weaker version of A.[1] If all tabbies are felines, then some tabbies are felines. The fact of the matter is that in the second case (AAI) we have drawn a weak conclusion when a strong one was possible. In the presentation of an argument, however, there is little point in not making the most of one's case. If you can produce premises which prove that *all* airships are dangerous, there is no reason to effect an anti-climax by concluding weakly that *some* airships are dangerous. It is good logic as well as human nature to make the most of an argument.

Consider another example, this time from the second figure:

A	All angels have wings.	A	All angels have wings.
E	No men have wings.	E	No men have wings.
E	Therefore, no men are angels.	O	Therefore, some men are not angels.

Again the same situation. When you come to examine the list of twenty-four carefully you will find that there are five combinations in which the full potentialities of the premises are not appreciated, and that in each case there is another argument in

[1] See p 115. In some cases this is doubtful

the same figure with identically the same premises but a stronger conclusion:

I		II	
AAI	(weak form of AAA)	EAO	(weak form of EAE)
EAO	(weak form of EAE)	AEO	(weak form of AEE)

IV

AEO (weak form of AEE)

These five combinations can without loss be dropped from the list and we are left with nineteen fundamental types of valid syllogistic argument. From two hundred and fifty-six to nineteen is quite a jump. We have sifted a lot of material. When you see the patterns into which the remaining nineteen arguments can be organized, you will realize that in spite of this drastic simplification of our problem there are amazing possibilities to examine.

A Christening Is in Order

These nineteen fundamental types have so long held a place in the study of logic that they have been given names, and to make the names easier to remember they have been incorporated into a Latin verse:

> *Barbara, Celarent, Darii, Ferio*que, prioris:
> *Cesare, Camestres, Festino, Baroco,* secundae:
> Tertia, *Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison,* habet:
> Quarta insuper addit *Bramantip, Camenes, Dimaris, Fesapo, Fresison.*

At first acquaintance it may seem that by going poetic we are just adding to our difficulties. But arrange the names under their figures, emphasize the vowels, and see what happens!

I	II
b A r b A r A	c E s A r E
c E l A r E n t	c A m E s t r E s
d A r I I	f E s t I n O
f E r I O	b A r O c O

III	IV
d A r A p t I	b r A m A n t I p
d I s A m I s	c A m E n E s
d A t I s I	d I m A r I s
f E l A p t O n	f E s A p O
b O c A r d O	f r E s I s O n
f E r I s O n	

Each name has only three vowels, the vowels are limited to A, E, I, and O, and they give in proper order the types of propositions that compose each argument. *Barbara* stands for the AAA argument in the first figure, *Festino* for the EIO argument in the second figure, and so on. As a matter of fact we have only begun to discover the significance of these names. Every little letter has a meaning all its own, as the song goes. This we shall see later.

Meet the Whole Family

At the moment, in order to familiarize ourselves with each of these nineteen arguments, it will probably be helpful for us to write down a fair example of each. In the first figure there are four: Barbara, Celarent, Darii and Ferio:

Barbara A All who are subject to passions can be fooled some of the time.
A All men are subject to passions.
A All men can be fooled some of the time.

(*Note:* If we were writing this in a strictly correct form it would be the following. The reader will want

to note this in all of the examples given. We render them in a less stilted form because they are more often encountered thus.)

All who are subject to passions *are included in* those who can be fooled some of the time.
All men *are included in* those who are subject to passions.

Therefore, all men *are included in* those who can be fooled some of the time.

Celarent E None who can reason can be fooled all of the time.
A All men can reason.
E No men can be fooled all of the time.

Darii A All who are careless in their thinking can be fooled some of the time.
I Some men are careless in their thinking.
I Some men can be fooled some of the time.

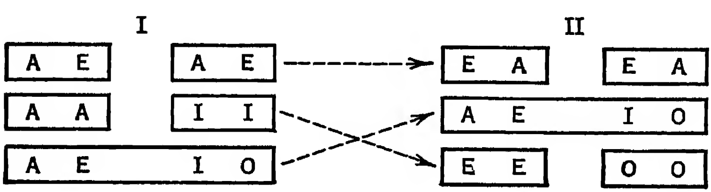
Ferio E None who are careful in their thinking can be fooled all of the time.
I Some men are careful in their thinking.
O Some men cannot be fooled all of the time.

There are several interesting uniformities to be observed in this first figure. One is that these arguments always begin either with an A or an E proposition; another is that the second premise is always either an A or an I proposition. Note also that the four conclusions take in all four possibilities in regular order; A, E, I, and O! There is a reason for this neatness, which the reader may be interested in working out for himself.

The second figure is equally neat and systematic. In it the first premises are also always A or E. But now the second premise runs the full AEIO gamut in the four combinations, and the conclusion is restricted to E and O propositions:

Cesare	E	Nothing that keeps perfect time is subject to irregular motion.
	A	All wrist watches are subject to irregular motion.
	E	No wrist watch keeps perfect time.
Camestres	A	All things that keep perfect time are stationary.
	E	No wrist watch is stationary.
	E	No wrist watch keeps perfect time.
Festino	E	Nothing that keeps perfect time has wooden works.
	I	Some grandfather's clocks have wooden works.
	O	Some grandfather's clocks do not keep perfect time.
Baroco	A	All things that keep perfect time are regulated.
	O	Some grandfather's clocks are not regulated.
	O	Some grandfather's clocks do not keep perfect time.

Not only are the internal relationships brought out, but the similarity of figures I and II is made evident by a diagram:



The more you look the more you see!

The third figure is the largest of all, containing six different arguments. The fact that the middle term occurs first in both premises makes it an easy one to use, hence with the exception of Barbara its combinations are most familiar:

Darapti	A	All who have hobbies are happy.
	A	All who have hobbies are men.
	I	Some men are happy.
Disamis	I	Some who have hobbies are happy.
	A	All who have hobbies are men.
	I	Some men are happy.
Datisi	A	All who have hobbies are happy.
	I	Some who have hobbies are men.
	I	Some men are happy.
Felapton	E	None who work in coal mines are happy.
	A	All who work in coal mines are men.
	O	Some men are not happy.
Bocardo	O	Some who work in coal mines are not happy.
	A	All who work in coal mines are men.
	O	Some men are not happy.
Ferison	E	None who work in coal mines are happy.
	I	Some who work in coal mines are men.
	O	Some men are not happy.

Neatness is also a characteristic of this figure. We have six arguments with which to deal. They can conveniently be broken into two groups of three, and if this is done a system emerges:

A	I	A	E	O	E
A	A	I	A	A	I
I	I	I	O	O	O

In the first three the major premises are A, I, and A; in the second three these premises are E, O, and E—the left and right sides respectively of the square of opposition. In the first three the minor premises are A, A, and I; and in the second group they are also A, A, and I. And, finally, the conclusions in the

first three are uniformly I propositions, while in the second group the conclusions are always O propositions. Notice that the diagram of the third figure resembles the diagram of the second figure. All of this takes an eye for structure, but by this time the eye should be keen!

One of the facts which you must have noticed is that Disamis and Datisi are variations on Darapti, and that Bocardo and Ferison have a similar relation to Felapton. We have purposely made the contents in each of these trios alike so that this will be apparent. Comparing Darapti and Disamis, for example, observe that the conclusion and one premise are exactly alike in the two and that the remaining premises are alike except that one (the I premise in Disamis) is a weaker version of the other (the A premise in Darapti). The only difference between the examples given for Darapti and Disamis is that in the former "*All* who have hobbies are happy" is the major premise, while in the latter it becomes "*Some* who have hobbies are happy." The latter premise says less than the former.

This should sound familiar. We were able to eliminate five syllogisms because they were weaker versions of the others on the list. In connection with the argument about tabby cats, we saw that if both a weak and a strong conclusion can be drawn from the same premises the arguments are essentially the same. Why not do some more eliminating here in the third figure? Disamis and Datisi are suspiciously like Darapti, and Bocardo and Ferison are suspiciously like Felapton. The curious fact we encounter here, however, is that a similar simplification is not possible. In this case the difference in strength is found in a *premise*, not in the conclusion. And this alters the situation completely. Disamis is not just a weaker version of Darapti, it is quite a different argument. This is one of logic's little surprises. If the same *conclusion* can be drawn from both a weak and a strong *premise* (in conjunction with another premise

which remains the same, the minor premise in this case) *it does not follow* that they are two versions of the same argument. They are separate arguments.

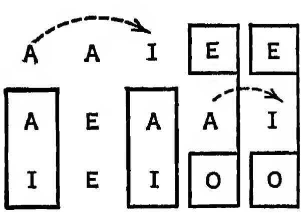
Perhaps the reason for this can be made clear in this way. An argument has a destination, is trying to get somewhere. A man on a desert dying of thirst is not interested in how far he can walk: he wants to know whether or not he can get to the oasis he sees ahead. "Can I get from where I am to where there is water?" It will cheer him to be told that he can walk a quarter of a mile beyond the oasis before he drops, but he will nevertheless stop to drink. Darapti and Disamis are like two men seeking water. They are standing together, but Darapti is stronger because he has more data at his disposal (i.e. he is employing the stronger premise): Disamis is weaker. But both reach the conclusion (the oasis). These are different problems. But AAA and AAI in the first figure represent two men of the same strength standing together (i.e. they have the same premises at their disposal); and if we can show that one of them can *more* than get to the oasis (establish a strong conclusion) there is no point in showing that the other can just make it (establish a weak conclusion). That is why the five arguments with weak conclusions were eliminated, but Disamis, Datisi, Bocardo and Ferison must remain.

Unhappily the fourth figure is the rowdiest of the quartet. It is not without some reason that Aristotle, who first worked on these relationships, refused to recognize the fourth figure as important in itself. He thought it simply a variation on the first. Indeed Bramantip and Camenes and Dimaris *are* like Barbara and Celarent and Darii with the slight difference that the conclusions of the latter three have their terms reversed. To-day the fourth figure is generally allowed to stand by itself, but it is notoriously messy and unsystematic, as can be immediately observed. In the first place, it has *five* combinations, a prime

number, in contrast with the even four and six of the other three. Five is a difficult number with which to work in a situation such as this one. The best we can do is to enumerate and illustrate the combinations, construct the diagram, and let the reader draw his own conclusions:

Bramantip	A	All who are graceful are young people.
	A	All young people dance.
	I	Some who dance are graceful.
Camenes	A	All who can see themselves are superhuman.
	E	None who are superhuman dance.
	E	None who dance can see themselves.
Dimaris	I	Some who are graceful are young people.
	A	All young people dance.
	I	Some who dance are graceful.
Fesapo	E	None who are graceful consciously seek popularity.
	A	All who consciously seek popularity dance.
	O	Some who dance are not graceful.
Fresison	E	None who are graceful consciously seek popularity.
	I	Some who consciously seek popularity dance.
	O	Some who dance are not graceful.

There is one element of system here that is worth mentioning, and it is the only one brought out by a diagram of this fourth figure:

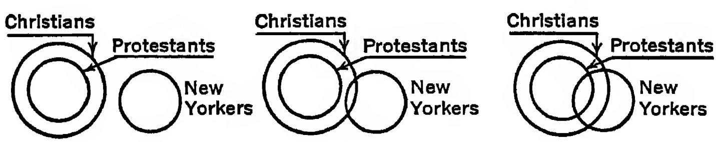


Camenes stands unattached, but it will be noticed that Bramantip and Dimaris, and Fesapo and Fresison form obvious pairs. In each pair the conclusion and one premise are exactly alike; and in each pair the other premises are alike except that one is a weaker version of the other. This situation is identical with that found in the weak and strong premises of pairs in the third figure, and hence needs no further discussion.

How to Recognize a Valid Syllogism When You Meet It in the Dark

What is the best way of knowing a valid syllogism from an invalid one? We meet them all the time. How test them? There are various ways in which it may be done. Certainly the simplest, though not the surest because of the human factor, is to draw diagrammatic circles such as the ones we have used by way of illustration from time to time. The circles are a sort of rule of thumb, easy to apply but treacherous because of the mistakes that can be made in constructing the figure.

As a matter of fact, even this rule of thumb sometimes involves complications. You must draw the picture that gives the widest interpretation of the premises. Suppose it is contended that some New Yorkers are not Protestants, because all Protestants are Christians and some New Yorkers are not. Confronted with this argument you can draw several diagrams before you have covered all possible cases and assured yourself that it is valid. Only the diagram on the right gives the widest interpretation:



Another method, almost as direct and more trustworthy when one gets accustomed to it, is to find out whether or not the argument in question can be given one of the nineteen names. For example: Some newspapers are untrustworthy, therefore this is untrustworthy because it is a newspaper. First we ask ourselves to what figure the argument belongs. Notice the arrangement of terms. This is a first figure argument. Then we ask what types of propositions make it up. In this case we have I and A and A. Is there any name in the first figure with the vowels I and A and A in that order? There is not. Therefore the argument is not valid.

Six Rules for Valid Syllogisms

There are many who prefer to do their testing by applying the six rules which have been developed to state all of the conditions which a valid syllogism must fulfil. It will pay to set down these rules. The reader does not need to be told that some invalid arguments are more insidious and dangerous than others. With the proper accompanying oratorical devices I can imagine getting away with this invalid argument: "Did you ever hear of a lover of his country who was not a law abiding citizen? Gentlemen, my friend here is a law abiding citizen, and therefore he must be a lover of his country." But there are some combinations which no one with the smallest amount of common sense would accept: "Some apples are red, and some tomatoes are red: therefore, some apples are tomatoes!" Rules are helpful in the more difficult cases.

Some of these six rules will be more useful than others. The first two are recommended for particular attention. Neglect of them is responsible for by far the largest number of fallacies in connection with the syllogistic type of argument:

Rule One. The middle term cannot be used twice undistributed,[1] *but must be distributed in at least one premise.*

The reason for this is clear. It is the function of the middle term to connect the other two in such a way that they may be joined in the conclusion. If in the major premise you refer to *some* of the middle term and in the minor premise you also refer to *some* of it, there is no way of being sure that in both cases you are speaking of the same portion of it. If you are told that all of the members of a secret organization are tattoed with a skull and crossbones, and then remember that some of your friends are similarly tattoed, you may be tempted to conclude that some of your friends are members of that secret organization. But you would be jumping to an unwarranted conclusion, for there is nothing to prevent there being those thus tattoed who do not belong to the organization. This example shows clearly how dangerously persuasive a fallacious syllogism may be. To break this first rule is to commit the *Fallacy of Undistributed Middle,* known to the inveterate punster as the Fallacy of the Middle-Aged Business Man.

Rule Two. If a term is distributed in the conclusion it must also be distributed in the premises.

If this rule is not observed one commits the *Fallacy of Illicit Distribution.* Call it the Speak-Easy Fallacy if you like to play with words. Persuasive but invalid arguments of this type also

[1] "Undistributed" and "distributed" refer respectively to whether *some* or *all* of the term is brought into question in the proposition. See pp. 51-2. The following diagram will refresh your memory of which terms in each of the four propositions are undistributed and which distributed

	Subject	*Predicate*
A	dist.	undist.
E	dist.	dist.
I	undist.	undist.
O	undist.	dist.

occur frequently. A speaker might argue thus: "You know, gentlemen, that all Communists believe in dictatorship. But surely I am no Communist. How, then, can you charge that I believe in dictatorship?" I cannot imagine a member of the audience rising to his feet and saying, "But look here, in your conclusion you use 'those who believe in dictatorship' to mean *all* who believe in dictatorship, but in your major premise you refer only to *some* who believe in it!" No mumblings about Illicit Distribution will get very far in a political rally, but there is probably no other situation in which they are more needed. In ordinary language this second rule simply says that you cannot validly include more of a class in your conclusion than is referred to in your premises. This is just common sense, or is it uncommon? No set of premises which mentions some monkeys will allow you to draw a conclusion about all monkeys. But, unfortunately, if you try it you will probably get away with it.

Rule Three. *No conclusion can be drawn from two premises in which the terms are connected by the relation of "excluded from"* (i.e. E or O).

Rule Four. *If the conclusion employs "excluded from" (i.e. is E or O) one of the premises must employ it also.*

Rule Five. *The conclusion always follows the weaker premise:* if one of the premises employs "excluded from" (i.e. is E or O) the conclusion must employ it also, and if one of them begins with "some" (i.e. is I or O) the conclusion must begin with "some."

Rule Six. *From two premises which begin with "some" (i.e. I or O) no conclusion can be drawn.*

We shall leave the reader to work out the reasons behind these last four rules, if he is interested. As the six rules stand they will be found useful in the testing of syllogistic arguments.

There is one other type of error into which the user of the syllogism may fall. The following argument is clearly fallacious: "All who dream are romancers, and therefore Jacob was a romancer because he was one who dreamed." This seems to be in the form Barbara and to obey all of the six rules we have laid down. Yet we know that something is wrong. What can it be? The truth is that this is not a syllogism at all, for it is a four-term and not a three-term argument. The point is that "dream" is used in two different senses: in the first premise it means "indulge in reveries" and in the second it means "entertain images during sleep." Obviously the two meanings do not coincide. Other cases are not so apparently erroneous. Do you see the fallacy in the following? "Man is a rational animal and Hitler is a man. Therefore Hitler is a rational animal."

This is generally spoken of as the *Four-Term Fallacy*, and occupies the middle ground between verbal and formal fallacies. Some of the world's craziest arguments are found in this region. Eaton gives two entertaining ones:[2] "Whatever is right is useful; only one of my hands is right; only one of my hands is useful." "Those who are unfitted to survive should be allowed to die; children who need protection against smallpox by vaccination are unfitted to survive; hence they should be allowed to die." Of course there is no reason why the fallacy need be limited to four terms. How many can you find in this pseudo-syllogism? "He has a heart of gold. Gold is heavy, and therefore his heart is heavy."

Conclusion

Now you have made the acquaintance of nineteen syllogistic arguments and can call them by name. It is like joining a new

[2] *General Logic* (Scribners), p. 100.

club, bewildering at first. So many new faces to remember! Train yourself to recognize these syllogisms in their native haunts. It is a good idea to use pencil and paper if you want to judge their characters and expose impostors with assurance.

Better still, get a feeling for structure as such. You can do exercises and apply rules until sunrise, but unless you get the "feel" of the thing you will have wasted your time. There are not many high school students who will deny that they can do the math assignment without understanding what it is all about. But this is not the way to become a good mathematician. And, too, you can do exercises in logic and apply its rules, name the correct syllogisms, and unearth suppressed premises, without understanding the structure behind these things. But if the high school student understands his mathematics, then he can use it for all it is worth and does not have that copy-cat dependence on the teacher's illustrative example with which we are all familiar when we get beyond our depths. The same holds for logic. If you understand, really understand, this system of nineteen arguments in four figures and appreciate its internal relationships, you will think more clearly than before.

Chapter Three

MORE ABOUT THE SYLLOGISM—IF YOU ARE INTERESTED

WHAT'S in a name? A rose by any other name might smell as sweet, but a syllogism by any other name would be far less exciting. Those nineteen queer words, from Barbara to Fresison, may seem like an invention of the devil when you try to memorize them but, believe it or not, they are designed to help you to understand the system of which they are a part. They are put together with an ingenuity which I hope will get your admiration before we are through. In the preceding chapter we described the syllogistic system of four figures and showed you the more obvious elements of its structure. But we have only begun to describe the pattern of the system. If you are interested in patterns you will find this more detailed description exciting; if not it will do no harm to skip to Chapter Four.

You may have noticed that the names of the nineteen valid syllogisms all begin with one of four letters, and that these four letters are the first four consonants in the alphabet, B, C, D, and F. You may also have noticed that all four initial consonants occur in the first figure; Barbara, Celarent, Darii, Ferio. Thereby hangs both an interesting tale and a new insight into the relations which these nineteen valid syllogisms have to each other.

Aristotle's Attempt at a System

Aristotle regarded the first figure of the syllogism to be more perfect than the other two (we remember that he did not recognize the fourth) and, having an eye for system, he wanted to show that the arguments in the second and third figures were dependent for their validity upon those in the first. He was interested in doing this because he believed that the arguments in the first figure clearly obeyed a principle which he considered to be basic to the syllogism, while the other arguments could be shown to obey this principle only after a certain amount of violence had been committed on them. His principle, in the Middle Ages called the *dictum de omni et nullo*, is variously rendered to-day. The following is a good version: *What can be predicated universally of anything else, whether affirmatively or negatively, can be predicated in the same way of all that is contained under it.*[1] Let us apply this to arguments in the first figure.

Barbara, Celarent, Darii and Ferio clearly obey this principle. The example given for Barbara was:

All who are subject to passions can be fooled some of the time.
All men are subject to passions.
____
All men can be fooled some of the time.

In this case it is predicated universally (that is, for all) of those who are subject to passions that they can be fooled some of the time. And since all men are contained under those who are subject to passions, it follows that all men can be fooled some of the time. Similarly with Celarent:

None who can reason can be fooled all of the time.
All men can reason.
____
No men can be fooled all of the time.

[1] Quoted from Eaton's *General Logic* (Scribners), p. 90.

Here it is predicated universally of those who can reason that they cannot be fooled all of the time. And since all men are contained under those who can reason, it follows that no men can be fooled all of the time. And Darii also:

All who are careless in their thinking can be fooled some of the time.
Some men are careless in their thinking.
Some men can be fooled some of the time.

It is predicated universally of those who are careless in their thinking that they can be fooled some of the time. And since some men are contained under the class of those who are careless in their thinking, it follows surely that these same men can be fooled some of the time. Hence some men *can* be fooled some of the time. Lastly, we have Ferio to consider:

None who are careful in their thinking can be fooled all of the time.
Some men are careful in their thinking.
Some men cannot be fooled all of the time.

We apply the formula of the principle once more. It is now predicated universally of those who are careful in their thinking that they cannot be fooled all of the time. And since some men are careful in their thinking, these men will not be fooled all of the time. Hence some men *cannot* be fooled all of the time. If we are willing to take the *dictum* as basic to syllogistic reasoning, and there seems to be no objection at the moment to doing so, we can rest assured as regards the first figure. It is "perfect": no need to worry about it.

The Logician's Law of Parsimony

But what about the other three figures? There are two courses open to us. Either we may take each of the arguments under

these figures off by itself and show after much manipulation that it falls under the *dictum*, or we may demonstrate that each of these other arguments is so related to an argument in the first figure that if the latter is valid the former must be also. The clue given you by the similarity of initial letters in the names leads you to believe that logicians take the second course. You are correct. Baroco, Bocardo and Bramantip are related to Barbara, and so on for the C's, D's and F's.

The reason for taking this second course is a good one, a logician's law of parsimony. The logician, and any one else for that matter, would be foolish to keep fifteen individual cases in mind if he can find a few general rules which cover them all. We do not try to remember that five apples plus three apples equal eight apples, that five pencils plus three pencils equal eight pencils, and so on. We simply remember that five plus three equals eight. In this case there are two general rules by which most arguments in the second, third, and fourth figures can be shown to be valid *if the corresponding argument in the first figure (the one with the same initial letter) is valid.*

S Stands for *Simpliciter:* One Route Back to the "Perfect" First Figure

You are beginning to realize that *all* of the letters in the nineteen latin names have meanings all their own. But we have just begun! You may have noticed that the letters "s," "p" and "m" are fairly liberally distributed among the names. You will also notice that they are absent from the names in the first figure. Now you have guessed it! They indicate the general rules by which the other fifteen arguments are reduced to the first four.

What does the "s" mean? Let us take an example with "s," Cesare, and see:

Nothing that keeps perfect time is subject to irregular motion.
All wrist watches are subject to irregular motion.
No wrist watch keeps perfect time.

Considering the initial letter, "C," this must have a definite relation to the first figure Celarent. The major premise is an E proposition. We learned in Chapter One that in an E proposition the terms may be reversed (the proposition converted) without changing the meaning of the original. So instead of saying "Nothing that keeps perfect time is subject to irregular motion" we may say "Nothing which is subject to irregular motion keeps perfect time." It makes no difference which we say. The two propositions are *equivalent* to one another. Since it makes no difference, suppose we substitute the second proposition in the argument in place of the first. This is what we get:

Nothing that is subject to irregular motion keeps perfect time.
All wrist watches are subject to irregular motion.
No wrist watch keeps perfect time.

Of course you see what has happened. We have transported the argument from the second figure to the *first*, for the middle term is now the *subject* of the major premise, and the argument is *Celarent.*

Now, since the major premises in these two cases are equivalent, and since the minor premises and conclusions are exactly alike, it follows that if one of the arguments can be shown to be valid then the other will automatically be valid also. The *dictum* tells us that Celarent is valid. Hence Cesare is also. It might be symbolized thus:

$$\begin{array}{lcl} \textit{Celarent} & & \textit{Cesare} \\ \text{All } M \not< P & \equiv & \text{All } P \not< M \\ \underline{\text{All } S < M} & = & \underline{\text{All } S < M} \\ \text{All } S \not< P & = & \text{All } S \not< P \end{array}$$

The "s" told us that Cesare could be reduced to Celarent by converting one of the propositions simpliciter which, we remember, is the way in which E propositions were converted. How did we know which to convert? We convert that proposition which precedes the "s" in the latin name, in this case the first E proposition. In Camenes we would convert the conclusion simpliciter.

P Stands for *Per Accidens:* Another Route to the Same Destination

By this time you are probably a step ahead. You have guessed that "p" stands for the other type of conversion, per accidens, and that a case like Darapti can be reduced to the first figure Darii by converting the second A proposition per accidens. Our example of Darapti was:

All who have hobbies are happy.
All who have hobbies are men.
____
Some men are happy.

The conversion of "All who have hobbies are men" *per accidens,* we will remember, will be the I proposition, "Some men have hobbies." Substituting this in the original argument, we are transported to the first figure Darii, as we guessed we should be:

All who have hobbies are happy.
Some men have hobbies.
____
Some men are happy.

But the situation here is slightly different from that first considered. "Some men have hobbies" is not *equivalent* to "All who have hobbies are men." We shall be safe, however, in saying that the former is *implied* by the latter. In other words,

if it is true that all who have hobbies are men, then it is true that some men have hobbies. The second is a weaker statement than the first.

And now comes a slightly difficult bit of reasoning. Follow closely. If we know that a ten pound blow will drive a nail through a board, we know that a twenty pound blow will at least do as much. If we can show that an argument with a weak premise is valid, then we have shown that another, which differs from the first only in that one of the premises is stronger than the corresponding one in the original, is also valid. Once more, the situation is similar to that of the men on the desert. Assuming that they are together but of unequal strength, if we can show that the weaker of the two (the one containing the weaker premise) has strength enough to reach the oasis (the conclusion), we have shown at the same time that the stronger can get there. The only difference between Darapti and Darii is that the second premise of the latter is weaker than that of the former:

Darapti				*Darii*		
All	M	P	$=$	All	M	P
All	M	S	$\supset$	Some	S	M
Some	S	P	$=$	Some	S	P

Since by the *dictum* we know that Darii is valid, the reduction *per accidens* has shown that Darapti is valid also.

M Warns Us of a Complication in Getting Back to Figure I

The interesting thing to note about the letter "m" in the latin names is that it occurs only when the *conclusion* of a syllogism has to be converted in order to reduce it to the first figure. This is a warning that when the conclusion is converted a situa-

tion of more than ordinary complexity arises. Consider the case of Dimaris:

Some who are graceful are young people.
All young people dance.

Some who dance are graceful.

The position of the "s" in the name tells us that the conclusion must be converted *simpliciter*, giving:

Some who are graceful are young people.
All young people dance.

Some who are graceful dance.

In this case *subject* and *predicate* terms in the conclusion are interchanged, which makes the original major premise the *minor* one, and the original minor premise the *major* one. Hence if the major premise is to come first the two must be interchanged thus:

All young people dance.
Some who are graceful are young people.

Some who are graceful dance.

And now we are back to Darii in its exact form. The "m" is a signal that the premises must be reversed after the proper conversion has been carried out.

A Summary of the Reductions Considered So Far

In some cases (e.g. Fresison) both premises have to be converted *simpliciter:* in others (e.g. Fesapo) one has to be converted *simpliciter* and the other *per accidens*. When all of the fifteen arguments in the second, third and fourth figures have

been examined, it is discovered that *nine* are *equivalent* to perfect arguments in the first figure:

Celarent	$\equiv$	*Cesare*	$\equiv$	*Camestres*	$\equiv$	*Camenes*
Darii	$\equiv$	*Datisi*	$\equiv$	*Disamis*	$\equiv$	*Dimaris*
Ferio	$\equiv$	*Festino*	$\equiv$	*Ferison*	$\equiv$	*Fresison*

In contrast, there are only *four* arguments requiring conversion *per accidens* and hence *implying* rather than being equivalent to the corresponding arguments in the first figure. They are the following:

Barbara			*Bramantip*	
All	$M < P$	$=$	All	$M < P$
All	$S < M$	$=$	All	$S < M$
All	$S < P$	$\supset$	Some	$P < S$
Darapti			*Darii*	
All	$M < P$	$=$	All	$M < P$
All	$M < S$	$\supset$	Some	$S < M$
Some	$S < P$	$=$	Some	$S < P$
Felapton			*Ferio*	
All	$M \nless P$	$=$	All	$M \nless P$
All	$M < S$	$\supset$	Some	$S < M$
Some	$S \nless P$	$=$	Some	$S \nless P$
Fesapo			*Ferio*	
All	$P \nless M$	$\equiv$	All	$M \nless P$
All	$M < S$	$\supset$	Some	$S < M$
Some	$S \nless P$	$=$	Some	$S \nless P$

The reader has undoubtedly observed one peculiarity of this diagram. Bramantip is on the *right* and the first figure argument to which it is being reduced is on the left: Darapti, Felapton and Fesapo are on the *left*, and the first figure arguments to which they are being reduced are on the right. Why should

this be? It looks off-hand as if we were "reducing" Barbara to the fourth figure rather than Bramantip to the first. I think it can be said without fear of exaggeration that if you can see the reason for this, you can rest assured that you have a sound understanding of what the syllogism is all about. It is one of those points which, when grasped, gives you the "feel" of a system.

It all goes back to those men on the desert. Let us suppose that a man of a certain strength and at a certain distance from the oasis can last long enough to reach it. In that case we know also that a stronger man at the same distance can reach the oasis. Darapti (also Felapton and Fesapo) may be looked upon as the strong man standing beside his weaker brother, Darii (Ferio, for Felapton and Fesapo). Both wish to reach the oasis *Some* $S < P$. The famous *dictum* tells us that Darii gets there, hence we can assure the stronger Darapti that he will get there too. In other words, the nature of the *premises* indicate the strength which the argument has and can employ to a given end.

In the case, however, of the sisters Barbara and Bramantip we find them standing together and of equal strength, which is a figurative way of saying that they employ exactly the same premises (except, of course, that they are reversed). But Barbara has a longer way to travel: she is trying to reach *All* $S < P$, whereas Bramantip is endeavoring only to get to *Some* $P < S$. In other words, the nature of the *conclusion* indicates the distance which the argument is trying to go with the strength at its disposal. You know yourself that if a man in Boston has enough gasoline in his car to drive to Philadelphia, he can also reach New York. And you know, likewise, that if your life depended on proving either *All* $S < P$ or *Some* $P < S$ you would choose the latter because it is less difficult.

Hence we may say that if the conclusion of one argument

implies the conclusion of another (assuming that they are otherwise the same) then the validity of the first implies the validity of the second; whereas if the premise of one argument implies the premise of another (assuming, again, that they are otherwise the same) then the validity of the *second* implies the validity of the *first.*

The Two Black Sheep of the Family, and the Meaning of the C in Their Names

If you have an ounce of sympathy you have begun to worry about Baroco and Bocardo. Where do they come in? They have neither "s" nor "p" in their names. And well you may worry. They are the black sheep of the family. They are neither equivalent to nor implied by any arguments in the first figure. Hence they cannot be reduced directly to "perfect" moods. If they could our system would have seemed much neater, but as a matter of fact their recalcitrance has led logicians to an analysis of the system at once neater and more profound than might otherwise have been suspected. It is an ill wind, they say. And at first it certainly *does* feel like an ill wind, for the only way in which Baroco and Bocardo can be brought into relation with Barbara is both indirect and complex. It took ingenuity to work it out the first time. The "c" in these two names indicates that the indirect method must be used. It might stand for "complex," but it does stand for "contradictionem," for the method employed in these otherwise non-coöperative examples involves the *contradiction* of the conclusion.

How do we get anywhere by contradicting the conclusion? The illustration we gave for Baroco was this:

All things that keep perfect time are regulated.
Some grandfather's clocks are not regulated.
___
Some grandfather's clocks do not keep perfect time.

It may seem pointless at the moment, but let us contradict the conclusion, changing it from its original O to an A, "All grandfather's clocks keep perfect time," and combine it with the original major premise to form a new argument:

All things that keep perfect time are regulated.
All grandfather's clocks keep perfect time.
___

This has the makings of the first figure Barbara and commands the conclusion, "All grandfather's clocks are regulated." But, and this is the important point, this new conclusion is the contradictory of the original *minor* premise! The same thing would happen if we combined the contradiction of the original conclusion with the second premise, except that in this case we should find ourselves developing the third figure Bocardo!

Some grandfather's clocks are not regulated.
All grandfather's clocks keep perfect time.
___
Some things that keep perfect time are not regulated.

And this also becomes Barbara when the conclusion is contradicted and added to the original A premise. Thus:

All things that keep perfect time are regulated.
All grandfather's clocks keep perfect time.
___
All grandfather's clocks are regulated.

From Baroco we get to Barbara either in a single step or via Bocardo. All three are thoroughly mixed one with another. There is something new and hitherto unsuspected here.

The Black Sheep Teach Us a New Principle

Why should this be? The general rule responsible for these interconnections is fairly simple when you get used to it. If

two premises lead *validly* to a certain conclusion, then if that conclusion is false *one* of the premises must be also. Let's get concrete. All stars twinkle: this is a star, therefore it twinkles. We accept this argument as valid. *Knowing that it is valid* we also know what to say if the light in question does *not* twinkle. Either (1) it is not a star, or (2) some stars do not twinkle. Note that either remark contradicts flatly one of the original premises. Another way of saying it is this. If we argue syllogistically to a conclusion, then discover that our conclusion is not supported by the facts, we may be sure either that the argument is invalid or that one of our premises is false. This sounds reasonable, does it not?

If as I approach home I smell an appetizing smell, I conclude that supper is ready. A savory smell indicates that supper is ready: I smell a savory smell. Ergo! Do I rush to the table? No. Being a logician, hence not entirely human, I want to test my reasoning. It would be easy just to walk into the kitchen and see. But logicians never do things directly. I can make the following test. I deliberately disregard my argument by sitting down to read the evening paper, indicating that I believe supper not to be ready. If I combine this judgment with the original one to the effect that a savory smell indicates that supper is ready, then I must necessarily come to the conclusion that I did *not* smell the savory smell. Or if I combine it with the original judgment that I did smell such a smell, then I must necessarily come to the conclusion that a savory smell does not indicate that supper is ready. If, in spite of my hunger, I can be sure of these last two easy-chair arguments, then I can be sure that my original argument was correct, for I have shown that *if* a savory smell indicates a ready supper and *if* I smell such a smell, then there can be no mistake. *Supper is ready*. I have wasted a few minutes in getting at the soup, but I have worked out an important principle.

Given any two propositions, P and Q, let us suppose that they imply a third proposition, R. Symbolically:

$$P \,.\, Q \supset R$$

This would be read, "P and Q implies R." The dot stands for "and" and the horseshoe for "implies." This will be recognized as a simplified version of any syllogism. P and Q are the premises, and together they imply the conclusion, R. Then *if P and Q implies R,* it follows that P and not-R implies not-Q. Remember the example of the star. If *all stars twinkle* (P) and *this is a star* (Q), then *this twinkles* (R). If *this does not twinkle* (not-R) and *all stars twinkle* (P), then *this is not a star* (not-Q). It is easy to see that P (all stars twinkle) and not-R (this does not twinkle) together could not imply Q (this is a star), for in that case P and Q would both be true and R would be false, which is contrary to our original statement that P and Q together *do* imply R. Stated in full, this principle becomes:

$$P \,.\, Q \supset R \quad = \quad P \,.\, \text{not-}R \supset \text{not-}Q \quad = \quad Q \,.\, \text{not-}R \supset \text{not-}P$$

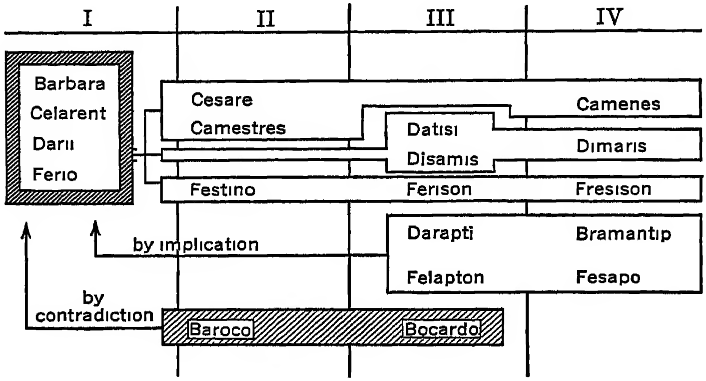
This, symbolically expressed, is the principle behind the indirect reduction of Bocardo and Baroco to the first figure. If

P stands for "All things that keep perfect time are regulated."
Q " " "Some grandfather's clocks are not regulated."
R " " "Some grandfather's clocks do not keep perfect time."

then Baroco would be symbolized as $P \,.\, Q \supset R$, and if we contradict the conclusion and add the result as a premise to P we get an argument which would be symbolized as $P \,.\, \textit{not-}R \supset \textit{not-}Q$. But we recognize the latter argument as Barbara and are assured by the *dictum* that it is valid. Hence we know that Baroco is also valid because the principle which we have employed states that they *equal* each other.

Summary of All Reductions

It has been a struggle, but it is worth the effort in the sense that we have now reduced all fifteen "imperfect" arguments to the four "perfect" ones. We may now make a diagram of the complete system:



Three methods have been employed: (1) equivalence, (2) implication, and (3) indirect reduction by contradiction.

A New Symbolism

But it has also been worth the effort in another sense. Baroco and Bocardo would be unmitigated nuisances were it not that in trying to reduce them we have discovered something of wide importance. Suppose we apply the new technique of contradicting the conclusion and adding it to one premise, to the four arguments in the first figure, which we know to be valid. In order to do so significantly we shall have to acquaint ourselves with a new type of symbolism.

Up to this point we have, for the sake of simplicity, used the signs for inclusion and exclusion in symbolizing A, E, I and O.

But instead of saying that "all S is included in P" we may say that "the class of things that are S's and not-P's is an empty class." If all apples are included in red things, then the class of things that are both apples and not-red is a class without members. It is the same either way. Using a dash (—) for "not" and a zero (0) for a class without members, the symbolism for an A proposition can be changed to:

$S - P = 0$ (things which are both S and not-P are equal to zero)

In the case of an I proposition, *Some S is P*, we are trying to say that things which are both S's and P's is *not* an empty class. If some apples are included in red things, then the class of things that are both apples and red is a class *with* members. This would be symbolized in the following manner:

$S\ P \neq 0$ (things which are both S and P are not equal to zero)

This new symbolism is in some respects easier to manipulate. For the sake of convenience we might put down a table showing the relation of the new symbols to the old. From what we have said the reader will understand what happens in the cases of E and O propositions:

A—All	$S < P$	$S - P = 0$	(things which are both S and not-P are equal to zero)
E—All	$S \not< P$	$S\ P = 0$	(things which are both S and P are equal to zero)
I—Some	$S < P$	$S\ P \neq 0$	(things which are both S and P are not equal to zero)
O—Some	$S \not< P$	$S - P \neq 0$	(things which are S and not-P are not equal to zero)

This new symbolism is also neater than the other. It shows clearly, for example, why E and I propositions may be converted *simpliciter*. E and I are the only ones in which S and P

occur with the same sign and hence may be written SP or PS indifferently. The reader will also observe that the equal sign indicates the *universality* of the proposition (*all* or *none*), while the inequality sign shows *particularity* (the class has *some* members).

The Importance of the New Principle. The Antilogism

We are now ready to go ahead. If we call a syllogism in which the conclusion has been contradicted an *antilogism*, we can set down in this new symbolism the four arguments in the first figure and their four corresponding antilogisms in the following table:

	Syllogism	*Antilogism*
Barbara	$M\,{-P} = 0$	$M\,{-P} = 0$
	$S\,{-M} = 0$	$S\,{-M} = 0$
	$S\,{-P} = 0$	$S\,{-P} \neq 0$
Celarent	$M\,P = 0$	$M\,P = 0$
	$S\,{-M} = 0$	$S\,{-M} = 0$
	$S\,P = 0$	$S\,P \neq 0$
Darii	$M\,{-P} = 0$	$M\,{-P} = 0$
	$S\,M \neq 0$	$S\,M \neq 0$
	$S\,P \neq 0$	$S\,P = 0$
Ferio	$M\,P = 0$	$M\,P = 0$
	$S\,M \neq 0$	$S\,M \neq 0$
	$S\,{-P} \neq 0$	$S\,{-P} = 0$

If you will examine the right-hand column carefully you will discover that something extraordinary has happened in working out the antilogisms for these arguments. Each antilogism has two equal signs and one inequality sign; each antilogism contains one term which appears once in each of the *equations* and

with a different sign in each; and the *inequation* in each antilogism contains the other two terms *with the same signs* they have in the *equations*. The order in which these elements appear differs from antilogism to antilogism but the structural elements are the same. This is most significant.

And it is not sheer accident, of course. The ambitious reader can figure out why these things have happened. They suggest, with a forcefulness which we may not neglect, the possibility that in working out the indirect reduction of Baroco and Bocardo (by contradicting the conclusion—i.e. changing the equality to an *inequality*, or vise versa) we have hit upon a general rule which applies to all valid syllogisms. We are reminded of the man who brought to a blacksmith five chains of three links each to get them made into a single chain. The blacksmith wanted to charge him for breaking and welding four links in the process, but he showed that it could be done (do you see how?) by breaking and welding only three. Similarly, we are on the threshold of a general rule for systematizing syllogistic arguments decidedly simpler than the one Aristotle and his followers had worked out.

We have seen that the more general a rule the more fundamentally it describes the structure involved. Instead of employing the *dictum* for Barbara, Celarent, Darii and Ferio; and reducing nine others to these by showing them to be equivalent; and reducing four more by showing that they are implied by the first four; and using the indirect method of reduction on Baroco and Bocardo—why not use the antilogism as a general rule covering all, as a sure test of the validity of *any* syllogistic argument?

Well, suppose we state the rules for the antilogism test and see what happens. *Contradict the conclusion of any syllogism, set up the resulting antilogism in symbols. If it obeys the following rules the syllogism is valid:*

1. There are two equations and one inequation.
2. The two equations have a term in common, which is once positive and once negative.
3. The inequation joins the other two terms, with the same signs they have when they appear in the equations.

This is both a more accurate and an easier test to apply than drawing circles or remembering names. We can be confident that if it holds, the syllogism in question is valid. The reader will do well to practise using it as a test of this type of argument. Familiarity with it is an asset in handling syllogisms.

The Nigger in the Woodpile

If the antilogism test is met, then the syllogism is valid. This we know. But look at it the other way around. If a syllogism is valid will it meet the test? Does this test hold for *all* valid syllogisms? This is quite a different question, and it uncovers the cutest little nigger in the logician's whole woodpile. We have seen that all four arguments in the first figure meet the test of the antilogism without flinching, indeed we used these four arguments as a way of introducing the test. We have seen also that the antilogism is a development arising out of the difficulty encountered in handling Baroco and Bocardo. It is a generalization of the method used in those two special cases, so it must hold for them.

So far we have accounted for six of the nineteen arguments. Thirteen to go! We remember that there are nine arguments distributed throughout the second, third and fourth figures which are equivalent to first figure arguments. They are Cesare, Camestres, Festino, Disamis, Datisi, Ferison, Camenes, Dimaris, Fresison, in case your memory is what mine is. It might be a good idea to set the reader to work testing these by the Antilogism test, but it will probably do no harm to tell what he can

guess from the fact that all nine are equivalent to first figure arguments, that all meet this test with flying colors. This is most encouraging. We have accounted for fifteen of the nineteen arguments by putting them under a single general principle. Quite an improvement. The antilogism was worth the effort expended in understanding it. Four to go, and we see the end in sight!

But do we? Alas, it is a mirage! Look at the antilogism of Bramantip if you want a surprise:

$$
\begin{array}{lcc}
 & \textit{Syllogism} & \textit{Antilogism} \\
\textit{Bramantip} & \begin{array}{c} P\ {-M} = O \\ M\ {-S}\ = O \\ \hline S\ \ P\ \neq O \end{array} & \begin{array}{c} P\ {-M} = O \\ M\ {-S}\ = O \\ \hline S\ \ P\ = O \end{array}
\end{array}
$$

Instead of having two equations and an inequation on the right we have three equations. The first rule is broken. And, what makes it worse, if any one of these *had* been an *in*equation the signs of the terms would still be wrong. This is one of those unexpected things that seem quite unnecessary. It would have been so neat otherwise! But in working out our system we just have to face the fact that Bramantip, Darapti, Felapton and Fesapo all violate the rules of the antilogism. Now *they* are the black sheep! We had trouble with them before, you remember. They already have a criminal record. When we were reducing the other figures to the first, they were the ones that could not be made equivalent; they were the ones for whom we had to go through that complicated process of implication. We remember them to our sorrow.

But hold on now. Let's not go too fast in convicting them. We *did* go through the process of implication, and we *did* show that if Barbara and Darii and Ferio were valid, these four must be also. Where is the fault? Is it in these "black sheep" or is it in the antilogism? There must be some reason why these four

do not conform. Suppose we put them side by side and see if we can observe anything unusual about them:

Bramantip	*Darapti*	*Felapton*	*Fesapo*
All $P < M$	All $M < P$	All $M \nless P$	All $P \nless M$
All $M < S$	All $M < S$	All $M < S$	All $M < S$
Some $S < P$	Some $S < P$	Some $S \nless P$	Some $S \nless P$

Just a glance shows what we are after. Each of these four has two universal premises (note the "all's"), and each has a particular (note the "some's") conclusion. Furthermore, these are the *only* ones of the nineteen arguments which have two universal premises and a particular conclusion. And, obviously, this is the reason why we get three equations in the antilogisms of these four. But what if they *do* draw a particular conclusion from universal premises. Is there anything wrong in this? If we draw circles for these syllogisms all seems to be well. We have seen that they can be reduced by implication to arguments which we know by every test to be valid. You will agree that we are faced with a curious situation.

What Can the Matter Be?

The explanation provides, I think, a fitting climax to the study of the syllogism. It is one of those things that are full of wonderment. The antilogism test seems to tell us that unless you employ "some" in the subject of one of your premises, you may not validly employ it in the subject of your conclusion. Barbara is the only other syllogism that does not have a particular premise—but it does not have a particular conclusion either. The key seems to be in the use of the word, "some." We arrive at the amazing conclusion that the study of the structure of a type of argument leads us to an unexpected refinement in the meanings of words. We are forced to the conclu-

sion that the word "some" contains a meaning absent from "all," and that if that meaning appears in the conclusion it must have appeared in the premises.

Let us examine some parallels in the use of these two words:

A—universal —All cats have claws.
I—particular—Some cats have claws.

Surely there is nothing startling yet. But consider a different parallel:

A—universal —All unicorns have horns.
I—particular—Some unicorns have horns.

If you are a stickler for accuracy you will see a difference here which makes A tenable but I untenable. When you say that all unicorns have horns you actually say: "All unicorns, *if there are any*, have horns." When you say that some unicorns have horns you actually say: "Some unicorns, *and there are some*, have horns." If you look the matter up in a dictionary you will see that this is correct. "Some" implies the *existence* of the individuals being discussed: "all" does not. You may speak of "some cats," but you may not speak of "some Cheshire cats." You may speak of "some elephants," but not of "some pink elephants." If your brother comes down to the breakfast table and describes last night's dream, saying: ". . . and some of the pink elephants wore straw hats": he is referring to elephants which had existence *in the dream*. You would let this next statement by without giving it a second thought: "All diamonds as large as my head are of incalculable value." But this would set you dreaming: "Some diamonds as large as my head are of incalculable value." The difference should be clear.

Fundamentally, the objection raised here by the antilogism tells us that if we are going to use words accurately we must revise our square of opposition. If an A proposition is true, it does not follow that the corresponding I one is also unless

the question of existence has been settled. Ditto E and O. Bramantip, Darapti, Felapton and Fesapo, which disregard the question of existence, are both valid and invalid, depending on the strictness with which we interpret validity. They are valid according to the *dictum*, but invalid according to the antilogism. Let us show how this new consideration works in a concrete case, Felapton:

> None who are in the front-line trenches are happy.
> All who are in the front-line trenches are men.
> ______________________________
> Some men are not happy.

The conclusion here is that there are men who are not happy. In the case of war, unfortunately, this argument is absolutely correct. If there *are* men in the front-line trenches, there are men who are unhappy. But in time of peace this argument is not correct, for the premises are just as true as ever while the conclusion *may* be entirely false, indeed it would surely be false if being in the front-line trenches were the only cause of unhappiness.

Curiously enough, the five redundant arguments which were originally struck from the twenty-four (1st fig. AAI and EAO; 2nd fig. EAO and AEO; 4th fig. AEO) are in exactly the same predicament. According to the *dictum* they are valid (but included in other valid arguments), while according to the antilogism they are as invalid as Bramantip, Darapti, Felapton and Fesapo. Hence they do not need to appear in a list of syllogisms in either case. In the one case they are redundant, and in the other they are invalid.

First we had two hundred and fifty-six arguments. These were quickly reduced to twenty-four. Of the twenty-four, nineteen are valid according to the *dictum* and deserve separate consideration (and names). But the principle unearthed in bringing Baroco and Bocardo under the *dictum* raised a new issue and

finally led to the conclusion that, all factors considered, only fifteen of the arguments may be treated as valid under every condition. We have run the gamut of the system of the syllogism. It is a rather extraordinary piece of structure, more complex than the square of opposition and in many ways more interesting.

A New Summary Based on the Antilogism Test

There are other elements of structure which we have not brought forward. One of these has to do with the antilogism. The fifteen arguments proved by the antilogism test fall into five groups of three, each group having a single antilogism. For example:

Antilogism

P All who have wings are angels.
Q No angels are human.
R Some human beings have wings.

Camenes

P All who have wings are angels.
Q No angels are human.
—R No human beings have wings.

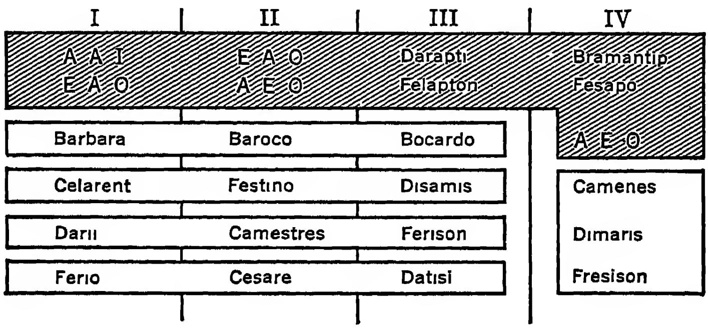
Fresison

Q No angels are human.
R Some human beings have wings.
—P Some who have wings are not angels.

Dimaris

R Some human beings have wings.
P All who have wings are angels.
—Q Some angels are human.

The reader may work out the other antilogisms for himself. The four groups into which the other arguments fall, indeed the whole second system of syllogistic argument uncovered by the antilogism, is made plain in the following diagram:



Close comparison of this with the diagram on page 108 will be profitable. In itself it is also worthy of study. A number of interesting comparisons are concealed in it. Notice, for example, that the triad which we illustrated was taken entirely from the fourth figure, and that the other four triads include an argument systematically selected from each of the first three figures. Some will want to know why this is so, and we leave them the problem and many other interesting points which we have not had time to touch—with our blessing!

We have tried throughout to stress the neatness and organization which emerge from a study of this kind. Here is the structure of thinking in some of its most fascinating details. We believe that if you have followed our points and become excited by the structure uncovered, as much as you would be excited by an intricate mosaic design in a Byzantine church, or by the interrelations of second-degree equations, or by the mechanism of a watch, or by the first movement of a Mozart quartet—then your thinking will be better.

Chapter Four

THE LOGICIAN'S MOLECULES AND HOW THEY ARE ARRANGED

YOU may at your pleasure analyze a glass of water into atoms or molecules. In either case it will quench your thirst. Similarly, you may analyze an argument into groups of terms, as in the case of the syllogism, or you may analyze it into groups of whole propositions. In either case it offers a challenge. Suppose we go back to that discomforting argument about seeing your dentist twice a year. We got our victim into the dentist chair, we hope, by taking three terms:

1. All who wish to reduce the pain of dentistry.
2. Those who should see the dentist at least twice a year.
3. The victim.

We called the first a major term, the second a middle term, and the last a minor term; and we put them together to form two premises and a conclusion according to a pattern known as the syllogism. Logicians have called this the atomic analysis of argument because each proposition contained therein is broken up into elements as different in character from propositions as gaseous hydrogen and oxygen are from liquid water.[1]

We are now about to consider what has been called the molecular analysis of argument, molecular because in it propo-

[1] The writer is indebted to Eaton for this use of "atomic" and "molecular."

sitions are considered as wholes. I could buttonhole my victim, for example, and present him with a molecular argument for going regularly to the dentist which logically is quite as effective as the atomic one suggested in the previous chapter. I could say: "Look here, old man, if you want to reduce the pain of dentistry, then you should see your dentist at least twice a year. And you do want to reduce the pain of dentistry, don't you? In that case you had certainly better make an appointment right away." Obviously the words which are structurally most important here are "if" and "then"; *if* one proposition is true *then* a second is also true. The course of the argument is entirely clear without breaking up the propositions of which it is composed:

> *If* (you want to reduce the pain of dentistry), *then* (you will see your dentist at least twice a year).
> (You want to reduce the pain of dentistry.)
> ____
> Therefore, (you will see your dentist at least twice a year).

The molecules, or propositions, of this argument are so related to each other that one of the molecules may be asserted as the conclusion of the argument. Using letters for the intact propositions and $\supset$ for "if . . . then" the molecular analysis becomes even more clear:

$$\begin{array}{l} P \supset Q \\ \underline{P} \\ \therefore Q \end{array}$$

This might be read: "If P then Q, and P is true. Therefore Q is true." It is called an *implicative* argument because the first premise involves the "implication" that if P is true then Q is true.

The general outline of this argument should not seem strange, even though we have switched from the terms M, S

and P to the complete propositions P and Q. Notice that it, like the syllogism, involves two premises and a conclusion. The line between the premises and the conclusion makes the similarity apparent. The difference is, of course, that while in the syllogism the premises are internally related to each other, in the implicative argument they are externally related. Or, in the syllogism the premises are atomically related, as one would relate hydrogen and oxygen to form water: in the implicative argument the premises are molecularly related, as one would relate oil and vinegar on a salad. In the former case something entirely novel is the result, a proposition which appears only in the conclusion: in the latter case, the conclusion appears as a whole in one of the premises.

The Implicative Arguments. From Four to Sixteen and Back to Four

In an implicative argument the first premise is always "P implies Q." The second premise makes an assertion about either P or Q as a whole. If we give attention to either of the two, P or Q, alone, it is plain that one may assert either that it is true or that it is false. Suppose we are considering Keats' assertion, "The poetry of earth is never dead." Either the poetry of earth dies or it does not. There is no middle path. Hence we may say of any proposition, P, that it is either true or false. And since we have two propositions, of either of which we may say that it is true or false, there are four possible premises which can be added to the initial premise that P implies Q:

$$\begin{array}{c} P \supset Q \\ \underline{P} \\ \end{array} \qquad \begin{array}{c} P \supset Q \\ \underline{-P} \\ \end{array} \qquad \begin{array}{c} P \supset Q \\ \underline{Q} \\ \end{array} \qquad \begin{array}{c} P \supset Q \\ \underline{-Q} \\ \end{array}$$

Thus there seem to be four possible implicative arguments. We may rejoice in being faced with a situation far simpler than that

offered by the two hundred and fifty-six in the case of the syllogism.

But are we rejoicing prematurely? Any one who is deliberately seeking complexity will point out that if all possible conclusions are considered the number of implicative arguments is increased from four to sixteen, because each of the four sets of premises may lead to one of four conclusions: For example, the first:

$$\begin{array}{cccc} P \supset Q & P \supset Q & P \supset Q & P \supset Q \\ P & P & P & P \\ \hline \therefore P & \therefore -P & \therefore -Q & \therefore Q \end{array}$$

But it is perfectly obvious that the first of these is *redundant:*

If this is a devil, then he will have a cloven hoof.
This is a devil.
Therefore, this is a devil!

The second and third are *self-contradictory:*

If this is a devil, then he will have a cloven hoof.
This is a devil.
Therefore, this is not a devil!

If this is a devil, then he will have a cloven hoof.
This is a devil.
Therefore, he will not have a cloven hoof!

In the first case you are building up an argument to prove what you are in a position to assert by itself as true. It is like taking a train from Kalamazoo to get to Kalamazoo. In the second case you are asserting that P is true in order to prove that it is not, like eating chocolates to get thin. In the third you are asserting that P implies Q and then arguing that it does not. No point in that either.

We are left, then, with the argument leading to the fourth conclusion. In this case, as in each of the other three sets of premises, there is but one conclusion that is plausible and at the same time not already stated directly above the line. Hence there are only four implicative arguments worthy of serious consideration. Each is based on one of the four sets of premises:

$$\begin{array}{ll} & P \supset Q \\ & P \\ \hline \therefore & Q \end{array} \qquad \begin{array}{ll} & P \supset Q \\ & -P \\ \hline \therefore & -Q \end{array} \qquad \begin{array}{ll} & P \supset Q \\ & Q \\ \hline \therefore & P \end{array} \qquad \begin{array}{ll} & P \supset Q \\ & -Q \\ \hline \therefore & -P \end{array}$$

Though the similarity with the syllogism is not close these are sometimes called the four "figures" of the implicative argument.

Two Are Good and Two Not So Good: Mind Your P's and Q's

When put in concrete form two of these structures, the first and last, are quite familiar. We all use them every day:

If I eat too much ice cream, then I shall be sick.
I eat too much ice cream.

Therefore, I shall be sick.

If I ate too much ice cream, then I shall be sick.
I shall not be sick.

Therefore, I did not eat too much ice cream.

The relentlessness of the gastronomic function in an emergency mirrors very memorably the relentlessness of these two logical structures. There is always the childhood experience of awaiting the inevitable after a secret spree. And there is always the envied pal with the cast-iron stomach who could not eat too much (from the point of view of violent consequences only, of course) of anything.

Childhood memories will probably also help illustrate that the other two implicative structures, the second and third, are invitingly plausible but quite unreliable. When you went to a party your mother probably warned you that if you ate too much ice cream you would be sick. And you innocently thought to avoid the sudden emergency by being very circumspect about the ice cream—and forgot to be cautious about the candies. Result: just what you had been trying to avoid. Hence the following structure, whatever its content, will be found invalid:

If I eat too much ice cream, then I shall be sick.
I do not eat too much ice cream.
Therefore, I shall not be sick.

and the person who uses it in this case will be an invalid invalid.

Of course Mother might also make a mistake in the matter if she is not careful. She was probably highly disgusted the evening you came home from a birthday party after being warned about eating too much ice cream, and promptly proceeded to be sick. "You naughty boy! Why did you eat so much ice cream!" If you were a child prodigy, and not too sick, you would sit up in bed and say: "Mother, you are being quite illogical about this. Our silly hostess made us play violent games too soon after supper, and I could not in all politeness refuse to participate." Hence the following structure also, whatever its content, is invalid:

If you ate too much ice cream, then you will be sick.
You are sick.
Therefore, you ate too much ice cream.

The two fallacies illustrated in these unfortunate cases are known respectively as *Denying the Antecedent* and *Affirming the Consequent*. In the former the antecedent of the two propositions connected by the implication in the first premise is denied

in the second. In the latter the consequent of the two propositions connected by the implication in the first premise is affirmed in the second.

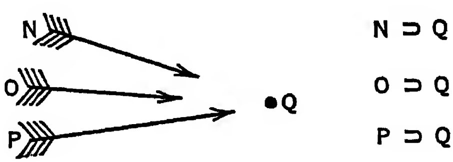
Of all of the fallacies considered by the logician these two most often occur in everyday discourse. Although the principle involved is simple, and is easily remembered once pointed out, the plausibility of arguments based on these fallacies is truly amazing. Consider the first. Remember the despair of the rejected Romeo who prophecied his own blissful happiness *if* he married Rosalind? Juliet showed him the fallacy of his conclusion that rejection by Rosalind must mean unhappiness. We are presented with so many if's in youth. If you do not do this you must stay after school. If you work hard you will be a rich man. One tends to forget that not all wealth is the reward of hard work. Teacher seems to mean that if you do *not* work hard you will *not* be a rich man. But there are lots of ways of getting rich. When you visit New York friends will tell you that if you go down to Greenwich Village you will see some queer people. On your return home you may report that you did not go to the Village, but it would be invalid to conclude therefrom that you had not seen some queer people!

And the other fallacy is quite as tricky. Your father has undoubtedly told you time and time again that if you drive carefully you will not get into accidents. Suppose that you do not get into accidents: your record is clear. Your father would be quite illogical to boast at the club that his son is a careful driver. You may be a reckless driver with either uncommon skill or unusual luck. The teacher who told you that if you worked hard you would be a rich man, might point quite erroneously to your large income tax return and say: "Here is a man who worked hard." Again, in the case of your trip to Greenwich Village, if you return home and report to your friends that you saw in the metropolis a lot of mighty queer people, it would be

quite illogical of them to unleash the gossip that you *had* been to Greenwich Village.

The point in the case of both of these fallacies is that the "If . . . then . . ." relationship between propositions, on which the entire implicative argument is based, does not mean "If *and only if* . . . then. . . ." When you say: If I eat too much ice cream then I shall be sick," you do not mean that eating too much ice cream is the only possible cause of illness. Gentlemen in love are not supposed to be logical, but Romeo and many after him would spare themselves much anguish if they realized that earthly bliss will be theirs *if* (not *if and only if*) they are accepted by one certain female. Perhaps teacher was right in saying that if you work hard you will be rich. These days the premise is doubtful, but surely she did not mean that you would be rich if *and only if* you worked hard. And surely no one with a sense of humor could state that you would see queer New Yorkers if *and only if* you encountered them in Greenwich Village.

It is because the relation of implication between prepositions does not mean "If and only if . . ." that you may not either assert that the second proposition is false if you deny the first, or assert the first as true if the second is affirmed. A diagram will make this clear:



As regards a given proposition, Q, there are several which imply it. For example, "The opposite sides of this figure are parallel" would be implied by, "This is a square," "This is a rhomboid," and "This is an octagon." But to deny that the

figure is a square does not permit one to deny that its opposite sides are parallel: nor may one infer that it is a square because its opposite sides are parallel.

Another Way of Putting the Molecules Together. The Disjunctive Argument

If there are two words which are more often used in argument than "If . . . then," they probably are the words "Either . . . or." Often you can get at your opponent by offering him an alternative and then leading him to the conclusion in which you are interested. "Either . . . or" is another way of joining the molecules of argumentation, and hence there is a second type of molecular argument to consider. It is called the *disjunctive* argument because in the premises the propositions are related to each other by a disjunctive relationship.

To return to the familiar problem of the advisability of making regular dentist appointments, we might approach our friend with these words. "Either you wish to increase the pain of dentistry or you will see your dentist at least twice a year. But surely you do not want more pain, hence you will do well to make that long overdue appointment." A more precise statement of this argument would be:

Either (you wish to increase the pain of dentistry) or (you will see your dentist at least twice a year).
It is false that (you wish to increase the pain of dentistry).
Therefore, (you will see your dentist at least twice a year).

The molecular character of this argument is clear when "v" is used as a symbol for "Either . . . or":

$$\begin{array}{rc} & P \vee Q \\ & -P \\ \hline \therefore & Q \end{array}$$

Either P is true or Q is true, and P is not true. Therefore Q is true. In the disjunctive argument the first premise is always "Either P or Q." And, as in the implicative argument, the second premise makes an assertion about either P or Q as a whole.

Symbolically the only difference between the two is that one employs a "$\supset$" in the first premise and the other employs a "v." Hence the same mathematical considerations as regards the number of possible arguments hold here that held in the case of the implicative argument. There are but four possible sets of premises. Also, as the reader may easily discover for himself, in the case of each of the sets of premises there is but one conclusion that is either interesting or plausible:

$$\begin{array}{cccc} P \vee Q & P \vee Q & P \vee Q & P \vee Q \\ \underline{P} & \underline{-P} & \underline{Q} & \underline{-Q} \\ \therefore -Q & \therefore Q & \therefore -P & \therefore P \end{array}$$

In this case also, then, we seem to have four figures to consider.

But there is a further simplification that can be made in dealing with disjunctions. "If . . . then" and "Either . . . or" are relations with different structural characteristics. The latter can be reversed without changing the meaning of the premise, while the former cannot. It makes no difference whether you say "Either he is in Boston or he is in New York" or "Either he is in New York or he is in Boston." But it makes a world of difference whether you say "If he lives in New York he reads the *New Yorker*" or "If he reads the *New Yorker* he lives in New York." $P \vee Q$ is the same thing as $Q \vee P$: but $P \supset Q$ is not the same as $Q \supset P$. Disjunction is a symmetrical relationship: implication is asymmetrical. Hence the first and third of the four disjunctive figures given above are essentially the same. In both the second premise asserts one of the propositions in question to be true (it does not matter which) and the conclusion asserts

the other to be false. And for similar reasons the second and fourth figures are fundamentally alike: both deny the truth of one of the propositions and conclude by asserting the truth of the other. Thus it is possible to reduce the number of significant disjunctive combinations to two.

What Do You Mean When You Say "Either . . . or . . ."? Two Possibilities

To revert again to memories of childhood, both disjunctive combinations are familiar and in frequent use. Was there ever a mother who did not use these arguments?

Either you stay in bed or you get a spanking.	$P \vee Q$
You do not stay in bed	$-P$
Therefore you get a spanking.	$\therefore\ Q$

Either you stay in bed or you get a spanking.	$P \vee Q$
You stay in bed.	P
Therefore you do not get a spanking.	$\therefore\ -Q$

Your mother presents you with a definite alternative, if she is a good disciplinarian. The poor disciplinarian is the parent who offers the child a definite pair of alternatives but does not stick to them in a crisis. You were, I hope, made to feel that staying in bed was security against a painful scene and that getting out of bed was a sure way to get punished. Of course you might well measure the pleasure of a few moments of freedom against the pain of the strap or hairbrush, but that was a different problem. The alternative was there and it was real. If your thinking was as clear as this you were a remarkable child, but you were thinking logically.

The day you rode your tricycle down the cellar stairs and scared your mother out of at least ten years of her life and your father said: "Either that child is incredibly dumb or

incredibly careless," you were faced with quite a different situation. You may have gone off into a corner to nurse a nasty welt on the forehead and comforted yourself with the realization that Father had said that you were dumb *or* careless. It was at any rate a consolation to know that you were not both! He had not said you were dumb *and* careless. But there you made a big mistake. It wasn't your fault of course. Your mother had used "Either . . . or . . ." in handling the getting-out-of-bed situation and had made it clear that a real alternative was offered. Your father had used "either . . . or . . ." also. But unfortunately grown-ups do not always mean the same thing when they use identical phrases. The chances are very good that if your father spoke with emotional stress he was entertaining the strong conviction that you were both dumb and careless.

Hence there is the possibility of a fallacy in the disjunctive argument. If you argued:

> Either I am incredibly dumb or I am incredibly careless.
> I am incredibly careless.
> ―――――――――――――――
> Therefore, I am not incredibly dumb.

you were arguing fallaciously. If you should assert that the book you were trying to read but could not seem to finish was either too long or was poorly written, your friend might pick up the book and discover that it was indeed more than a thousand pages in length and conclude that, since it was so long, you failed to finish it because it was long and not because of its style. That would obviously be poor reasoning. This fallacy does not occur often, and its consequences are not usually serious, but it does raise an interesting point.

In legal documents you often encounter the curious "and/or." This is the lawyer's way of being sure he says what he means. The situation is similar to that faced with the implicative argument. It all depends on what you mean by the words you use,

from a structural point of view. When you say, "If . . . then . . ." you never commit a fallacy so long as you mean "If *and only if* . . . then . . ." When you say, "Either . . . or" you never commit a fallacy so long as you mean "Either . . . or . . . *and not both.*" When your mother warned you about getting out of bed she was assuring you that either you stayed in bed or you got a spanking *and not both.* But your father's reference to the trip down the cellar stairs did not exclude both possibilities as regards your character.

The fallacies of *denying the antecedent* and *affirming the consequent* are important because we seldom mean "and only if" when we use "If . . . then. . . ." But when we employ "Either . . . or . . ." we almost always have in mind what may be called a *strong*, or an *exclusive, disjunction.* Either he is in New York or he is in Chicago. He is in New York; therefore he is not in Chicago. Or, he is not in New York; therefore he is in Chicago. He cannot be both places at once, and his presence or absence in one place tells all that you need to know about his presence or absence in the other. This is a strong disjunction. But when after an uncomfortable night you say that either the lobster was not fresh or you exercised too violently after eating it, you are asserting a *weak* disjunction.

For reasons which will become clear when we discuss deductive systems, logicians have agreed to let "$\supset$" stand for the plain "If . . . then . . ." implication, and "v" stand for the weak disjunction. Hence when symbolically expressed only *two* of the implicative arguments are valid, and only *one* of the disjunctive arguments is. This must necessarily be so, as one can easily see. When one is confronted only by symbols all possibility of considering context is removed. And since those two implicative arguments and the one disjunctive argument are *always* valid no matter what the content, we may safely accept them. When the other forms of these arguments appear the only thing we can do is

to examine the context to find out which structure is given to "If . . . then . . ." and which to "Either . . . or. . . ." It may be that the argument is nevertheless valid. It will be if you can legitimately insert "and only if" or "and not both." Otherwise it will be fallacious. But one must be able to examine the concrete argument to determine this. And, obviously, the concrete argument cannot be reduced to symbols because the logical symbol is made possible only by the removal of concrete meanings.

Three More Intricate Implicative and Disjunctive Arguments

There is a variety of more difficult molecular arguments found by using more complex implicative or disjunctive premises. Some involve more than one implication or more than one disjunction, and some are ways of combining the two. For example, a useful form of argument may be composed with two implicative premises. If Johnny takes a bath then the bathroom gets soaked: and if this is Saturday night, then Johnny takes a bath. Hence if this is Saturday night, then the bathroom gets soaked:

$$\begin{array}{rc} & Q \supset R \\ & P \supset Q \\ \hline \therefore & P \supset R \end{array}$$

The resemblance of this structure to that of our friend Barbara, the syllogism, is striking. It is also worth thinking about. There is also a direct relation between this and the valid form of the simple implicative argument in which the antecedent is affirmed. The reader can for himself work out a second valid complex implicative argument (i.e. employing two implicative premises) based on the simple argument in which the consequent is denied.

Is it also possible to combine two disjunctive premises? Yes,

under certain circumstances. If you were confronted with the assertion that a certain student is either a senior or a sophomore because he was seen at a Senior-Sophomore smoker, and that he is either a senior or a graduate student because he was seen in cap and gown at Commencement, there is a conclusion that can be drawn *if the meanings of the "either . . . or . . ." 's involved are carefully considered.* Perhaps the reader can express symbolically the structure of such an argument.

But of all of the complex molecular arguments the most famous is the dilemma, of debating fame. If a friend tells you he is in a dilemma, he means that he is in trouble and cannot find his way out. To confront your opponent in argument with a dilemma is like charging him, head down, nostrils breathing fire. He may be rooted to the spot with fear, in which case he and his argument are lost. The speaker who does not meet the challenge of a dilemma can have little hope of victory. The speaker who knows how to meet a dilemma may, like the skilful toreador, live to fight another day.

Now just what is a dilemma? It is an argument in which one premise is an *implicative* proposition and the other is a *disjunctive* proposition. There is the story of the Greek father who was trying to persuade his son not to go into political life. His argument will be quoted as long as dilemmas are known. First he confronted his son with a premise involving two implications. "If you go along with the majority, then you will be unhappy because in the wrong. And if you go along with the minority, then you will be unhappy because unsuccessful":

$$P \supset Q \text{ and } R \supset S$$

Then he confronted his son with a premise involving a disjunction containing some of the same propositions: "But either you must go along with the majority or with the minority":

$$P \vee R$$

The conclusion was obvious. Put these two premises together, draw a line, and you have a very powerful argument:

$$\begin{array}{ll} & P \supset Q \text{ and } R \supset S \\ & \underline{P \vee R} \\ \therefore & Q \vee S \end{array}$$

"Therefore you will either be unhappy because wrong, or unhappy because unsuccessful." The boy must necessarily be unhappy if he goes into politics. The argument seems completely devastating.

Of course what the father is trying to show is that whatever the boy does in politics he will be unhappy. Hence the father might have simplified his argument and just said: "If you go with the majority you will be unhappy, and if you go with the minority you will be unhappy. But you must do one or the other. Hence you will necessarily be unhappy in politics." This would allow a simpler analysis of the dilemma:

$$\begin{array}{ll} & P \supset Q \text{ and } R \supset Q \\ & \underline{P \vee R} \\ \therefore & Q \vee Q \end{array}$$

Psychologically this is more effective, for by concluding "Q v Q" you are saying in effect, "The same thing happens in either case." Hence it is characteristic of most dilemmas to be stated in this form. But a glance at the symbols composing each shows that logically they are much alike. This latter simple form is actually a special case of the former. In discussing the dilemma, then, we shall use the former symbolism. It is more general and contains the latter within it.

Three Ways of Meeting the Charge of an Onrushing Dilemma

Should you find yourself in a bull-ring with an onrushing bull there are several things you might do in order to save yourself. If you are a man of Herculean strength you might *take the bull by the horns* and throw him to the ground. And one of the things the Greek boy might have done would have been to take his father's dilemma by the horns. He might say: "No, father, you are wrong. If I go along with the majority I shall be unhappy only at first. In time I shall lead the majority around to a more correct way of thinking. I shall teach the majority to want what is right." In doing this he would be denying one of his father's implications:

$$
\begin{array}{ll}
 & P \supset T \\
 & \cancel{P \supset Q} \text{ and } R \supset S \\
 & \underline{P \vee R} \\
\therefore & \cancel{Q \vee S} \quad T \vee S
\end{array}
$$

P does not imply Q, it implies T. Hence a different conclusion is possible. If he went with the majority the boy might find happiness by showing them the right paths.

The skilful toreador meets the charge of the bull in another fashion. He jumps aside and lets the bull rush past without touching him. And, similarly, the nimble debater will find another way of avoiding the effect of his opponent's dilemma. This method is known as *slipping between* (or around) *the horns* of the argument. This is accomplished by denying the disjunctive premise, by showing that there are other alternatives than those offered. Again, the Greek boy might say to his father: "You are wrong. I do not have to go along either with the majority or the minority. I can pay enough attention to the majority to stay in office, and at the same time enough attention

to the minority to progress in the right direction. The situation is more complex than you picture it to be."

$$\begin{array}{l} P \supset Q \text{ and } R \supset S \\ \underline{P \vee R \mathit{v} T} \\ \therefore\ Q \vee S \mathit{v} T \end{array}$$

In this case a different conclusion is made possible by the addition to the second premise of the alternative T. The boy might find happiness by going along with both the majority and the minority.

There is a famous Baron Munchausen story about the predicament in which the Baron once found himself with a wolf rushing at him when he was completely unarmed. What did he do? He waited until the wolf was right on top of him with mouth wide open. Then he reached his hand in the mouth, past the tonsils, down the gullet, through the stomach and on until he could grab the wolf's tail, jerked suddenly and mightily on the tail thus turning the wolf inside out and sending him off in the other direction. I should not recommend this as a technique for the average bull fighter: the bull is long and the human arm is short. But it works beautifully against the onrushing dilemma. Turn it inside out! The Greek boy had an enviable opportunity to answer his father by turning his argument back on the father in the form of another dilemma. "If I go along with the majority then I shall be happy because successful. And if I go along with the minority I shall be happy because in the right. You are quite right in saying that I must do one of the two. Hence I should go into politics because I shall inevitably be happy." The second halves of the implications in the initial premise are negated and interchanged:

$$\begin{array}{l} P \supset \overset{-S}{\cancel{Q}} \text{ and } R \supset \overset{-Q}{\cancel{S}} \\ \underline{P \vee R} \\ \therefore\ \cancel{Q \vee S}\quad -S \vee -Q \end{array}$$

The conclusion has been made to go in precisely the opposite direction!

This is technically known as the *rebuttal* of a dilemma. Its effectiveness is logically accurate and psychologically overwhelming. Beware of facing your opponent with a dilemma that can be rebutted! There are many famous rebuttals. Protagoras once trained an Athenian lawyer with the understanding that he should be paid for his instruction after his first successful case. When the payment was delayed because Eulathus was slow in starting his legal practice Protagoras sued him for the amount. In court Protagoras argued that if he won the case Eulathus should pay because of the judgment of the court; and that if he lost Eulathus should pay because he had agreed to pay as soon as he had won a case. It looked bad for Eulathus, but he produced a grand rebuttal. What was it?

There is something too neat about a dilemma. In formal argument it has a deserved place, but in everyday life it may be a menace. I had a room-mate in college, a mathematics major, who used playfully to confront me with dilemmas to persuade me to his way of thinking. Until I knew enough logic to realize how they could dress up a situation by making the issues too clear, I was the goat. Think how oppressed you can become by arguing to yourself in this way: "If I stay up late to study for this examination, then I shall flunk because of lack of sleep. Yet if I do not stay up late, then I shall flunk because I do not know the material." Another chance here for effective rebuttal. It should always be remembered that while psychologically extremely compelling, the dilemma, like any other argument, is no more powerful than its weakest premise. It has a domineering personality and will go around playing Achilles, but unlike Achilles it is just as vulnerable as any other mortal argument. No need to fear it if you keep your wits about you.

Destructive and Constructive Dilemmas

You may get the impression that all dilemmas are highly destructive. So they are, if you are not careful. But curiously enough the ones we have been examining are called *constructive* dilemmas. They are so named because in all of them the disjunctive premise has affirmed the truth of the antecedent of one of the implications.

From our study of the simple implicative argument it is clear that another type of dilemma might in its second premise deny the consequent of one of the implications:

$$\begin{array}{ll} & P \supset Q \text{ and } R \supset S \\ & \underline{-Q \vee -S} \\ \therefore & -P \vee -R \end{array}$$

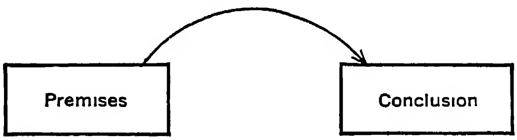
If he comes to trial he is not a clever politician: if he comes to trial and is found guilty he does not have a clever lawyer. But either he is a clever politician or he has a clever lawyer. Therefore he will either not come to trial or he will not be found guilty. This is a *destructive* dilemma. The constructive and destructive dilemmas correspond to the two valid figures of the implicative argument. Other examples of the latter are readily found. In its simpler and more effective form this argument would be:

If he is punished he is not a clever politician, and if he is punished his lawyer is not clever.	$P \supset Q$ and $P \supset R$
But either he is a clever politician or his lawyer is clever.	$-Q \vee -R$
∴ In neither case will he be punished.	$\therefore -P \vee -P$

This, like the simpler argument which the Greek father might have used to his son, is a special case.

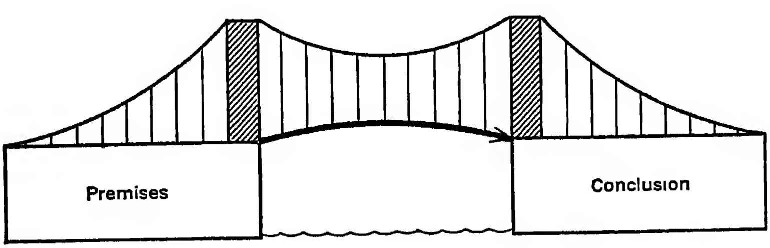
Deductive Arguments Are Like Bridges

All of the arguments considered so far have been deductive arguments, that is they have been ways of proceeding from a set of premises to a conclusion, thus:

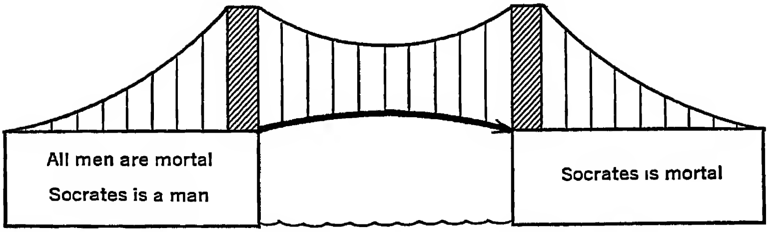


The argument carries you over from premises which you are in a position to assert as true, to a conclusion which you are interested in proving. The validity of the argument, its *logical* value, is determined by whether or not the premises have the structural strength to carry you over to the conclusion.

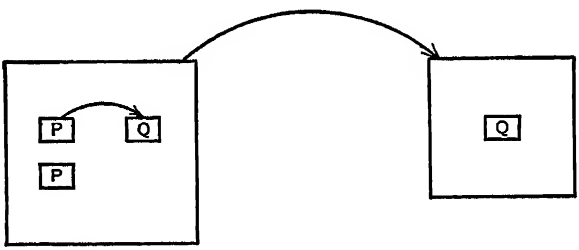
In some ways it is as if you were an engineer building a bridge across a river. Has the structure of the edifice you are building enough strength to carry you across the river?



From this point of view it is clear that *all* deductive arguments are large implications. *If* the premises are true, *then* the conclusion is true. In the syllogism, for example, *if all men are mortal and Socrates is a man, then Socrates is mortal.*



And what we have called the implicative argument is really one which contains two implicative relationships, a large one and a small one. The large one is the implicative relationship between the premises as such and the conclusion, and the small one is a relationship within one of the premises. Thus:

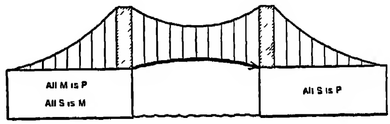


In addition to offering one type of molecular argument, the implicative relation is fundamental to all deduction of conclusions from premises.

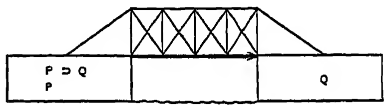
If you are interested in getting across a river there are many types of bridge you can build. And if you are interested in reaching a certain conclusion there are many valid structures that will take you there. We have examined a few of the more common ones. We may argue either molecularly (implicatively or disjunctively) or atomically (syllogistically) that Socrates is mortal, for example. The molecular arguments are simpler (note the difference in the number of figures) and hence appear more often in ordinary discourse. But the two types are fun-

damentally the same: *if* the premises are true, *then* the conclusion is true. The metaphor of the bridge holds for all three:

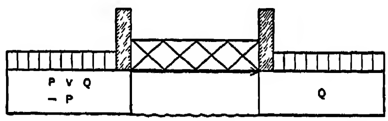
All men are mortal.
Socrates is a man.
Socrates is mortal.



If Socrates is a man, then he is mortal.
Socrates is a man.
Socrates is mortal.



Either Socrates is not a man, or Socrates is mortal.
Socrates is a man.
Socrates is mortal.



Any of the simpler deductive arguments can be expressed syllogistically, implicatively or disjunctively. The reader may amuse himself by reinterpreting the arguments of Chapter Two in the other two structures. Sometimes one analysis will suit a given argument better than another, but essentially the three analyses are interchangeable.

One of the interesting things to note in passing is that while the three arguments above about poor mortal Socrates represent different structures, they are surprisingly alike. In all three cases the second premises and the conclusions are identical. The whole difference is found in the first premises. Put these three first premises together and see what happens.

(1) All men are mortal.
(2) If Socrates is a man, then he is mortal.
(3) Either Socrates is not a man, or Socrates is mortal.

All three say the same thing in different ways. They all express

the same idea but according to different structural relationships. It is like this situation in arithmetic:

$$8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 = 64$$
$$8 \times 8 = 64$$
$$8^2 = 64$$

We leave the reader to work out this idea further for himself.

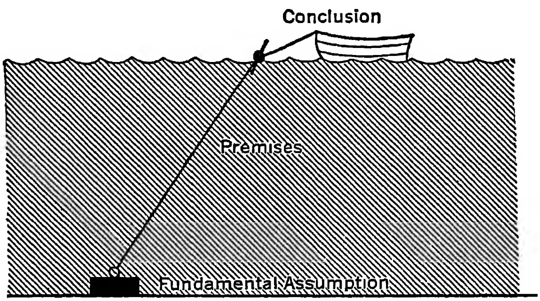
The Importance of Deduction and the Problem of Anchoring a Boat

Deductive arguments, whether atomically or molecularly organized, offer certainties, or proofs. If our temporizing friend will admit the truth of the premises about going to the dentist, then the truth of the conclusion that he should immediately make an appointment follows with certainty. Once the premises of a valid deductive argument are accepted the conclusion must also necessarily be accepted.

But, as we have seen in the case of the dilemma, premises may be open to question. The problem of whether a premise is *true* or not is a concrete question involving content and must be decided in individual cases. Does my friend wish to avoid the pain of dentistry? Perhaps not. Perhaps he is a sadist. Perhaps he would rather spend his time and his money carving statues. But the problem of whether a deductive argument is *valid* or not is an abstract question involving structure and has an answer universally acceptable. If I can corner my friend and get him to accept the premises I offer, then I can if he is rational get him to accept the conclusion which necessarily follows from these premises.

Structural analysis, whether atomic or molecular, allows us to probe for the fundamentals of our thinking. Do you consider it wrong to steal? *Why?* Try to find more general premises

which will lead necessarily to that conclusion. Would you take up arms in case of war? *Why?* Do you believe in the existence of God? *Why?* What is the importance of a liberal education? *Why?* Our everyday judgments are like objects floating on the surface of water. Are they anchored to a rock bottom? Are they anchored in poor holding-ground which will let the anchor drag as soon as any pull is exerted? Or are they unanchored and drifting in and out with the tide? When you come to tie up your boat it makes a lot of difference what, if anything, is below the surface. And it makes a lot of difference what premises are below the surface of our everyday thinking:



It would be a poor sailor who did not examine his moorings regularly. And it is a poor thinker who does not probe his underlying premises.

The analyses we have been discussing are the instruments of such a probe. Perhaps everything is O.K. below the surface. More probably not. Some of our fundamental assumptions look mighty silly when hauled to the surface and examined. My friend of the dentist argument might try to put me off with an argument based on the premise that if his teeth do not hurt they are healthy enough. But the slightest knowledge of biology shows that to be a poor anchorage. Others of our fundamental assumptions when brought up are in an awful tangle. Hosts of beginning students believe, (1) Only tangible things are real,

(2) The soul is real, (3) The soul is intangible. They do not put it as clearly as this, but below the surface of their everyday thinking they are in a jungle of contradictions. And, finally, there are many underlying propositions to which conclusions are never drawn. It was not so long ago that many would have agreed that all who hold property should have the ballot and who knew that women held property, yet voted against woman suffrage! These unconnected premises are like cables caught below the surface.

Deduction is important. We have only begun to show its importance. It is also fascinating in the variety and neatness of its structures. We have only begun to tell its story. Before going further we must tell the story of another type of structure.

PART TWO

The Structure of Experimental Procedure

Chapter Five [1]

THE DETECTIVE AND HIS ART

A MURDER has been committed. A hurried call to police headquarters: officers rush to the scene of the crime: everything is left untouched until the circumstantial evidence can be examined. A good beginning for any detective novel. Then comes the inevitable rivalry between the police and our hero, the private detective. The body is seated slumped over a desk, gun in hand, shot in the temple. The door was locked on the inside, a statement of serious financial losses lies on the desk, nothing disturbed in the room to denote a struggle, the only window closed and locked. The inevitable police verdict—suicide. But the family knows some inside story about the dead man, cannot believe he took his own life, calls in the detective. Our hero begins to act queerly, scoops up cigar ashes, paces the room, examines locks on doors and windows minutely, takes pictures of the corpse, reads the watermark on the financial sheet. You know the familiar situation: his keen eye and open mind have shown him something that does not fit with the suicide hypothesis. Perhaps the gun is in the right hand and the wound on the left temple: that *would* make monkeys out of the police. Perhaps the victim's pocket handkerchief is missing and the police are unaware that he invariably carried one.

[1] For the main outline of this material the writer is indebted to Castell, *A College Logic* (Macmillan).

The problem of every detective novel is the problem of developing an hypothesis to account for facts. The better the book the more facts at the disposal of the reader, enabling him to be his own detective. The usual formula is for all of the facts to recommend one hypothesis strongly during most of the book, and then for some sudden turn in events to make it necessary to throw that hypothesis aside in favor of another. All suspicions point to one unfortunate individual whose alibi is bad and whose every motion is an admission of guilt. The trap is set — and sprung — and in it is found the victim's wife, who had been above suspicion and who had been boldly coöperating with the police at every point in the investigation! The better the novel the more the turns and surprises, the more various the possibilities of interpreting the facts, the more hypotheses worth investigating, and the more the characters under suspicion.

Deduction and Induction Compared: the Law of Gravity

Every detective novel is a treatise in logic, far more interesting than most, and possessing all of the elements of an important type of structure in our thinking. The major difference, and a most important one, between a detective novel and the work of a flesh-and-blood detective is that when we finish the novel we know with finality the solution of the mystery, while the flesh-and-blood detective can never know whether or not he has found the correct answer. In other words, no detective ever *proves* his case: the best he can succeed in doing is to make it appear highly *probable.* The logic of developing hypotheses is a logic of probabilities, never of certainties.

In deduction the conclusion is necessarily and certainly true if the premises are accepted, as in the case of my friend and his dentist appointments. But in induction, the logic of inducing

an hypothesis from the facts, the hypothesis is never more than probable when the facts have been accepted. For example, no one will ever know with certainty whether or not Hauptmann was guilty of kidnapping the Lindbergh baby. At the famous trial the attorney for the defense did an extraordinarily clever thing in arranging all of the facts in support of the hypothesis that it was Jafsie who did the kidnapping! Any one who wants a good example of how the most complete set of facts may be marshalled in more than one direction will find the defense's summation of that case highly interesting. The jury found the state's hypothesis more *probable* than that of the defense, but there never was *proof* that the man who was punished for the crime committed it. This is true of any case at law that has to be decided on circumstantial evidence.

Deduction and induction move in opposite directions. And the difference between the two makes it seem as if there were in operation a logician's law of gravity. It is easier to drop a pebble from a third-story window on the head of some one below than it is to throw a pebble from the ground against a third-story window. In one case gravity helps, in the other it hinders. Deduction moves downward from the general to the particular: induction moves upward from the particular to the general:

If you can stand above the facts and assert the premise that all men are mortal, it is not difficult to drop a pebble down on the individual Socrates and prove that he is mortal. But if you are

standing on the ground watching Socrates and Plato and Thrasymachus and Antyus and the others, it is impossible to toss a pebble which will make a bull's-eye against a target up in the air which generalizes to the effect that all men are mortal. To return to the court of law, a supreme court can hand down a final judgment in the interpretation of a law. Is censorship a violation of the rights of the individual guaranteed by the constitution? The Court might argue thus: "All cases in which freedom of speech is taken away are violations of the rights of the individual. Censorship is a case in which freedom of speech is taken away. *Ergo*." But a court which is dealing with the individual facts of a crime and trying to arrive at some generalization about what actually happened may only give out a probable judgment.

One more point of difference. The actual content of a proposition is decidedly important in induction. We have seen that the validity of a deductive structure is completely unconnected with the content of the propositions in that structure. If any two judgments of a given structure (say *All M is P* and *All S is M*) are placed in company with one another, *any* conclusion with a certain structure containing those terms (*All S is P*) will be valid. This essentially is the reason why deduction deals with proofs. In induction we put facts together in a structural relationship, as we shall see, but it does not follow that *any* facts placed in that same structural relationship will lead to the same probability in the hypothesis. Probability is not probable in general: one must always say, "It is probable *on the basis of certain facts* that. . . ."

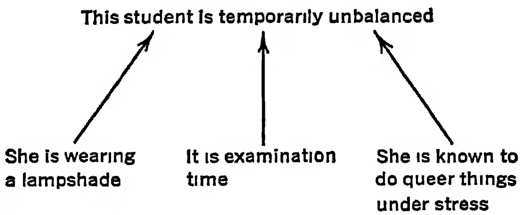
If you were trying to persuade a jury that a prisoner was at fault in an automobile accident, the fact that he was seen just before the accident traveling at eighty miles an hour down the wrong side of the road would be much more valuable than the fact that he had a record for traffic violations. On the other

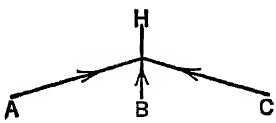
hand, if you were trying to persuade an insurance agent that this man should not be accepted in a mutual company the fact of a record for traffic violations would be more valuable than the fact of his once being seen traveling at such high speed where he did not belong. In other words, while in our symbolization of the structure of developing an hypothesis we shall place the facts A and B and C in a certain definite relationship to one another, their significance in the structure will depend on what they individually stand for and not just upon their being facts.

A good poker player will calculate the probabilities of his drawing a fifth card to complete a straight against his chances of completing a flush. There are only four outstanding cards that will fill the straight and there are nine outstanding that will fill the flush. That, of course, is why a straight counts more in the final reckoning. Unless the cards are stacked the poker player can never be certain what card he will draw. He can only develop an hypothesis based on probabilities. The gambler who uses loaded dice is able to develop hypotheses of higher probability than those of his opponent, but even he loses money sometimes. In playing the game of ordinary life the cards are not stacked and the dice are not loaded. We must examine the known facts and face the future armed with hypotheses, but we must remember the hypothetical character of our judgments. Students sometimes toss a coin to decide how the evening shall be spent. Heads we go to a movie: tails we go to the dance: if the coin stands on edge we study. Some day the coin *may* stand on edge!

Convergence: The Famous Case of the Lampshades

I am walking along on campus when I see in front of me a student wearing a lampshade on her head. Coupling this with the fact that it is about examination time, I conclude that some poor unfortunate has become temporarily unbalanced. She does not seem to be violent, so I nod politely as if I had not noticed anything unusual and pass on. Two facts have converged on a satisfactory hypothesis, satisfactory because I remember that this particular student has been known to do queer things under stress. Hence:



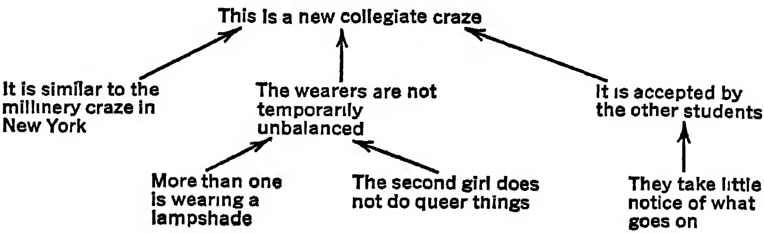


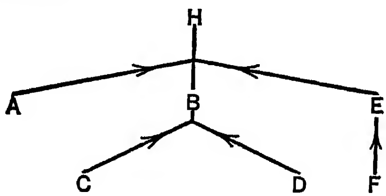
Note here that the symbolic structure of this convergence gives meaning to my thinking but nevertheless tells only part of the story. The rest of the story is contained in the individual facts. We cannot substitute any fact for "B," for example. The fact that the student's name is "Smythe" would have infinitesimal bearing on my hypothesis.

Well, I go on my way, concerned but not frightened. Suddenly I am aware of another student approaching, also wearing a lampshade as headgear! In all common sense this new fact completely destroys the probability of my first explanation of

these extraordinary actions. Thus does the logician's law of gravity work. One fact which fits into the pattern of a given hypothesis adds slightly to the probability that it is true. One fact which refuses to fit into the pattern reduces the probability to zero. It is like climbing up a greased pole. Every good hitch upward gets you nearer the prize, but one slip and you slide rapidly to the bottom again.

This second apparition has certainly sent me to the bottom of my pole again. How explain *two* girls so adorned? Excited now by keen curiosity, I search in my mind for another hypothesis upon which the first three and this additional fact will converge. Perhaps some other fact, stored in memory, will aid in developing a new hypothesis. I recollect, by an obvious process of association, the new fashion in hats which caused me to gape the last time I was on Fifth Avenue. Surely these "lampshades" are no more queer than many of the styles I witnessed. Perhaps this is a new collegiate craze. Then I notice that other girls on campus are taking little notice of these curious headgears, which also supports my second rather than my first hypothesis. Thus I have developed the following convergent structure:





This is obviously an advance over the earlier hypothesis. One fact, "It is examination time," has dropped out as irrelevant, but more facts are placed in the structure. My interpretation of the situation is more adequate.

An important principle in this type of structure now appears. In developing an hypothesis the facts are not lined up in single file, like soldiers on parade, all equal in significance and interchangeable. They are organized according to a pattern that begins to look like the branchings of a tree. Some facts belong together, because they converge along the same line upon the hypothesis. For example, "More than one is wearing a lampshade" and "The second girl does not do queer things" co-operate in making possible one major step toward the hypothesis, "The wearers are not temporarily unbalanced." Other facts are quite unrelated except as they both have a part in the general convergence. Similarly, on a tree certain branches are joined to the same limb. Cut that limb and both branches die. Other branches are quite unrelated except as they are branches of the same tree and die if the whole tree dies. This comparison of the structure of a convergence to the structure of a tree is excellent—except that you have to invert the tree, pointing the branches downward. And it is easy to remember because five of the letters in "conv*ergen*ce," all right together, spell "g-r-e-e-n," a color generally associated with trees.

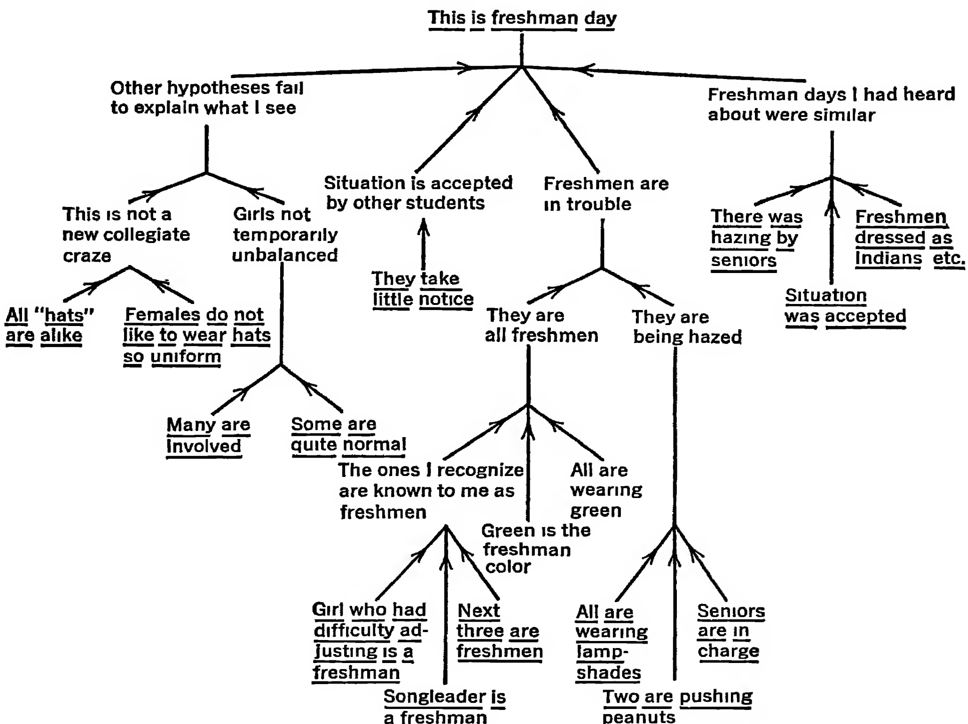
After seeing the second girl wearing the lampshade I go on my way, relieved that the first girl is still balanced in mind, but somewhat troubled by the aesthetic taste of the female of the species. I encounter a third, a fourth, a fifth girl so adorned—and begin to smell a rat. I suddenly realize that there is a uniformity about this extraordinary phenomenon, entirely foreign to the usual female custom in selecting hats. Girls do not by preference wear hats that look like so many others! Here is a fact about female psychology that does not fit into the con-

vergence. It becomes necessary to find a third and more satisfactory hypothesis that will explain all this. More facts are needed!

Examining these "hats" more carefully I discover that they are *really* lampshades. Furthermore each is tied to the head with a *green* ribbon. Why the green ribbon? Why green? Three of these girls approaching at the moment are in my freshman class; and I remember that the first one I encountered was a freshman who had difficulty adjusting herself to college work, and that the second was freshman song leader. These perambulating floor lamps are all freshmen! As I round one of the buildings I see two of these freshmen pushing peanuts along with their noses under the care of a couple of hard-boiled seniors. I recollect things I have heard about the trials and tribulations of freshmen. A new hypothesis flashes to mind, and immediately all of the facts work together into a much more convincing structure than any I had yet devised. *This is Freshman Day*. The diagram which can be made out of these facts looks even more like the structure of a tree than the previous one (See Fig. I, page 156).

The Structure of a Convergence Examined

Now that we have worked out what may be called a full-fledged convergence we shall do well to stop and examine it. Notice first of all that this does not constitute a *proof* that it is Freshman Day. With every fact noted and fitted into the structure the probability that my hypothesis is correct is increased, but still I may be as thoroughly wrong now as I was in building up the first two hypotheses. It is a healthy and fruitful logical exercise to take the various facts in a convergence and marshal them so that they converge on *another* hypothesis. This is always possible. It is the best demonstration that induction deals



This is freshman day
Other hypotheses fail to explain what I see
Situation is accepted by other students
Freshmen are in trouble
Freshman days I had heard about were similar
This is not a new collegiate craze
Girls not temporarily unbalanced
They take little notice
They are all freshmen
They are being hazed
There was hazing by seniors
Situation was accepted
Freshmen dressed as Indians etc.
All "hats" are alike
Females do not like to wear hats so uniform
Many are involved
Some are quite normal
The ones I recognize are known to me as freshmen
Green is the freshman color
All are wearing green
Girl who had difficulty adjusting is a freshman
Songleader is a freshman
Next three are freshmen
All are wearing lamp-shades
Two are pushing peanuts
Seniors are in charge

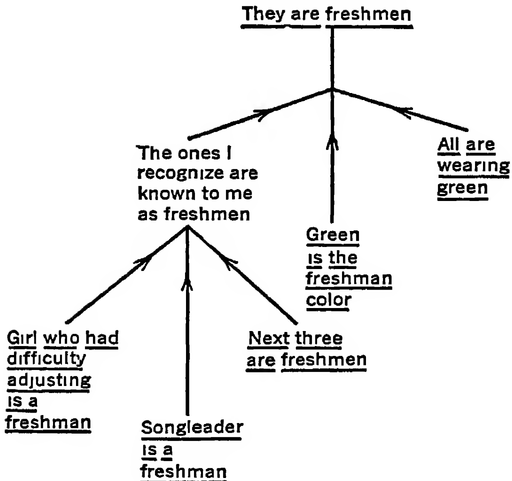
FIGURE I

with probabilities not proofs. In the case of the lampshades, what other convergence could you construct from the facts? Suppose you took as your hypothesis that this is the day when certain students are elected to sororities? The facts could be made to fit that hypothesis quite as well. Can you think of another hypothesis that would explain the facts? If you are ingenious you will be able to think of several.

Not only will the same set of facts converge on more than one hypothesis, but they can be made to converge in more than one manner on a given hypothesis. There is no unique structure according to which the facts are marshalled. Two detectives working on the same case, equipped with the same set of facts, may easily arrive at the same solution of the mystery by way of different paths. A further exercise, of your ingenuity would be to take the facts given in the convergence on page 156 and rearrange them in a different manner to reach the same conclusion about Freshman Day. Try it, and compare the two diagrams.

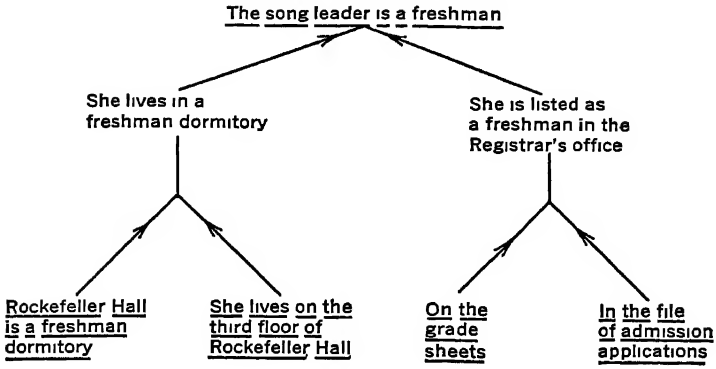
Perhaps the most important thing to notice about the structure of a convergence is that it is composed of three elements: (1) a major hypothesis, (2) sub-hypotheses, and (3) the facts of the case. If we underline the major hypothesis and the facts of the case (as on p. 156) all of the elements in between are sub-hypotheses. If we follow the branches downward as far as they will go we inevitably come upon a fact: if we follow the branches upward we come inevitably to the major hypothesis toward which the entire structure is pointing. One might suppose that the only hypothetical element in a convergence is the proposition which we are endeavoring to make probable. But no. Everything between the facts and the major hypothesis is also hypothetical.

Suppose we take one little section of the convergence on p. 156 as an illustration:



This is in itself a complete convergence, and if it stood alone the "They are freshmen" would be called a major hypothesis. But in Figure I it is only a sub-hypothesis which contributes something to the probability of the major hypothesis about Freshman Day. Hence it may be said that every convergence is a hierarchy of convergences; every convergence is made up of a number of smaller convergences. We do not *know* that all in question are freshmen: this is just an hypothesis which helps support the larger hypothesis in which we are interested. Whenever we come upon a proposition that may be accepted without question we can underline it and treat it as a fact, and that particular part of the convergence ends at that point.

How do we know when to stop? A very important question. If we want to be meticulous in our investigation we can go much further than in Figure I. In that figure we have accepted "The song leader is a freshman" as a fact. But how do we know? We might treat this as an hypothesis and marshal facts in support of it thus:



And why stop here? Why not treat "Rockefeller Hall is a freshman dormitory" as another sub-hypothesis? Well, the answer is that life is short. Sometimes we have to see with our own eyes in order to believe, as for example the sight of these girls actually wearing lampshades. At others we may safely accept statements on authority. If some one tells me that Miss Fortescue is freshman song leader I am fairly safe in believing that she is. We cannot go about investigating everything we hear. Hence in a convergence we shall treat as a fact any proposition that meets common sense tests. My wealth of experience with women tells me that they are not entirely happy when they see other women wearing hats very much like theirs, hence I am going to treat that as a "fact" without writing a doctor's dissertation on *The Psychology of the Female of the Species Homo Sapiens with Special Reference to Millinery*.

Sherlock Holmes and the Prediction Technique

Sherlock Holmes was a lover of the dramatic. He often led Scotland Yard and the reader to the criminal by anticipating what would happen next and getting to the scene of action ahead of time. No one who has read *The Speckled Band* will

forget Holmes and Dr. Watson waiting in the dark for a killer snake to crawl through a transom. In *The Red-Headed League* the great detective was able to predict an attempt at robbing a bank. Then there was *The Case of the Six Napoleons*, in which he amazed the owner of a cheap bust of Napoleon by buying it from him for ten pounds and calmly proceeding to smash it to bits—because the hypothesis he had formulated told him to expect to find an oriental pearl inside! Needless to say, he found it.

But in all of the Sherlock Holmes stories there is no scene more famous than that in *The Study in Scarlet*, when to the bewilderment of all present Holmes fed to a decrepit dog milk containing part of one of two pills that had been found at the scene of the second murder. As usual the two Scotland Yard detectives had gone off on false scents. Holmes had studied the scene of the first crime microscopically, had described the appearance of the murderer minutely and even told what kind of cigar he smoked, and seen almost immediately that the victim had been poisoned. But how? With what? Holmes made the private prediction that an instrument of poison, probably pills, would be found. So when Lestrade mentioned casually that two pills were found at the scene of the second crime, Holmes cried, "The last link. My case is complete!" And with an air of mystery and expectation he fed part of one to the dog. *Nothing happened.* Holmes gnawed his lip and drummed his fingers on the table. The two detectives "smiled derisively." Suddenly with a perfect shriek of delight Holmes rushed to the box, prepared part of the *other* pill with milk and fed it to the old dog. In Conan Doyle's own words: "The unfortunate creature's tongue seemed hardly to have been moistened in it before it gave a convulsive shiver in every limb and lay as rigid and lifeless as if it had been struck by lightning."

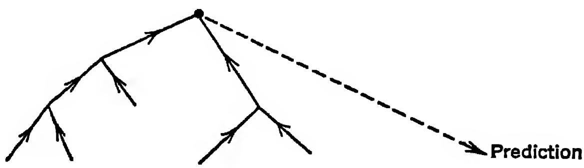
Here is an elaboration of the convergence technique. The facts are studied and placed in a converging structure supporting a certain hypotheses. Then the hypothesis is assumed to be correct and a predication is made regarding a *new* fact. If the predication is fulfilled the hypothesis is accepted. This happens often in the best detective stories. The hero predicts that a second murder will be attempted at a given spot and at a given hour; the chief characters rush to that spot just in time, or just too late, to prevent another tragedy. We feel secure that the detective who is able to predict future events successfully is on the right track. He usually is. In that Famous Case of the Lampshades, I could also formulate a prediction. *If* to-day is Freshman Day all of the rest of the freshmen I meet will be wearing lampshades tied on with green ribbon. This turns out to be the case, and I dismiss the mystery from my mind as solved.

The Psychology and Logic of Prediction Are Quite Different

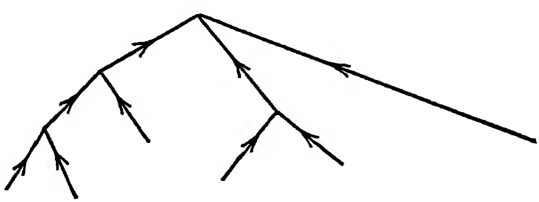
There is nothing more convincing than a prediction fulfilled. From a certain wildness in the eye, increasingly frequent dates, innumerable letters, and a secretly packed suitcase, you formulate the hypothesis that your daughter is going to elope this evening. Circumstances are such that *if* your hypothesis is true she will have to go by way of the window to avoid detection, and if she is going by way of the window she will have to use a ladder. You stick your head out of the window and there is the ladder! You are convinced. The more unusual the prediction the greater the effect. If you are told that upon walking four miles into the woods you will find at the base of a certain tree a concert grand piano, you are impressed. As between a simple convergence and a convergence plus a predic-

tion we should choose the latter every time. It seems to imply a true understanding of the facts.

But now comes one of the great surprises in logic. A prediction is dangerously misleading in the conviction it lends to a convergence. Its effect is almost entirely psychological, and logic reduces it to the status of a very small incident in the development of an hypothesis. Using a dotted line to indicate a prediction, the case of your daughter would symbolically be represented in the following manner. Note carefully the directions of the arrows:



But suppose for the moment that when you looked out of the window you had not yet formulated an hypothesis about your daughter's actions. Suppose you were merely mystified in general and had gone to the window to look at the moon as a way of forgetting your worries about her. You see the ladder, and you suddenly fit this fact in with the others to formulate an hypothesis. Your convergence would then look like this:



Note that it is exactly like the one involving a prediction except that the dotted line is now continuous and its arrow is reversed. *A prediction only adds one more fact to the convergence.* The major hypothesis is no more probable as a result of the predic-

tion than it would have been had the predicted fact been part of the original data. But prediction is dramatic, is emotionally appealing, and that is why it is employed so often in detective stories. The element of prediction *in itself* adds nothing to the probabilities of the case. A little clear thinking will show this to be so in the Sherlock Holmes story and the case of the lampshades.

Don't Look Now, but the Argument from Prediction Is Mighty Suspicious

Perhaps the most interesting thing about the method of prediction is that it seems to involve a fallacy. *If* Sherlock Holmes' hypothesis is correct, *then* these pills will be found to be poisonous. These pills are poisonous, therefore his hypothesis is correct. Symbolize this molecularly and see what you get!

$$\begin{array}{rl} & P \supset Q \\ & \underline{Q} \\ \therefore & P \end{array}$$

Does this look familiar? It is the fallacy of Affirming the Consequent. Sherlock Holmes' hypothesis *might* be correct and the pills have nothing to do with the case. The other freshmen I meet may be wearing lampshades tied with green ribbon, yet I *might* be meeting only freshmen chosen for a certain sorority. If there is a ladder at the window when you look out, it *might* have been placed there by a burglar. These possibilities being open and the argument from prediction involving a fallacy, must we not conclude that we are arguing invalidly when we employ prediction? The answer to this question tells the whole story of the difference between induction and deduction, and is worth studying with extreme care.

It would obviously be more difficult to *prove* that a man had

committed a murder than it would be to make it appear *probable* that he had done so. Deduction attempts proof: induction seeks probabilities. Remember the man on the desert seeking an oasis. Evidence in the form of a footprint that he had been able to get half way to his objective would add legitimately to the probability that he got to the oasis, but would not prove that he got there. In other words, the requirements for proof are more rigid than those for probability. The process of Affirming the Consequent does not meet the rigid requirements for proof, but it does meet the less rigid ones for the establishment of a probability. If I know that if I eat to much ice cream I shall be sick, the fact that I am sick does not prove that I ate too much ice cream *but it does make it more probable than I did so. If I were not sick* it would be proof that I did not eat too much ice cream: *if it were not known whether or not I was sick* there would be no fact on which to base either a probability or a proof: *if I am sick* a probability is established. The reader will do well to ponder these three cases.

Our Hero and the Local Constabulary: the Method of *Reductio Ad Absurdum*

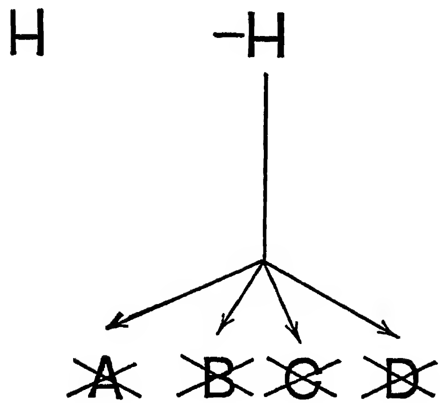
Most detective story writers have a neat way of demonstrating the limitations of the local constabulary, that of showing in great detail the absurdity of the hypothesis on which it is working. And whatever works against the police detective works in favor of our hero. Showing that other hypotheses are absurd is one way of increasing the probability that the one in which you are interested is correct.

In one of the later Sherlock Holmes stories, *The Mystery of Thor Bridge*, a jealous wife plants a revolver with one chamber fired in the wardrobe of the governess, goes down at night to a bridge, ties a heavy rock with cord to another revolver of the

same caliber and suspends the rock over the parapet of the bridge; then, holding in her left hand a note written by the governess confirming a midnight appointment with the wife, she shoots herself. The revolver is carried over the parapet into the river, and when the body is found the case looks to the police very much like a murder. That is, to the local constabulary it looks very much like murder.

But good old Sherlock Holmes sees the absurdity of the murder hypothesis. What would be the point of carrying in the hand a note which there was no need to consult? Absurd! Why should the governess, having been seen near the bridge, walk home calmly and go to bed? If the act was premeditated she would have a better alibi: if the act was not premeditated why would she be carrying a gun? Again, absurd. And, finally, why should she carry the gun back to the house with her and put it in her wardrobe where it would surely be found, when it would have been so much easier to throw it into the marshy ground beside the river where it would never be discovered? Likewise absurd. Sherlock Holmes could easily enumerate the unsurmountable difficulties in the path of believing that the woman had been murdered by the governess. Ergo, she must have killed herself with the intention of implicating the unfortunate girl.

The principle involved here is the logician's law of gravity, of which we spoke earlier. It takes only one fact that will not fit into a structure of convergence to show that the major hypothesis is questionable, whereas a whole host of acceptable facts can do no more than make it probable. Since it is so much easier to be destructive, suppose we become destructive in a big way, taking care to destroy the *contradictory* of the hypothesis in which we are actually interested. If we can show the contradictory to be absurd, then we have made a strong case for the original hypothesis. Thus:



This method is based on a structure which looks very much like that of a convergence. But note that now all of the arrows are pointed downward. What we actually have in this case is a *divergence* rather than a convergence. If the hypothesis is true, then it is probable that such and such facts are the case. *If* the woman was murdered by the governess, then it is probable that the former had no need for the note she held so conspicuously and that the latter did something very foolish with the gun. If your daughter is not going to elope, then it is probable that some one has in the dead of night put a ladder at the window for no reason. It is often easy in cases like these to make your opponent look very foolish. You meet him on his own ground and show him to be an ass.

But, as a matter of logic, how strong is a case built by the *reductio ad absurdum* method? We assume an hypothesis to be true, arrange in a divergence the facts implied by it, show that these facts are absurd (hence false) and thus that the hypothesis is false:

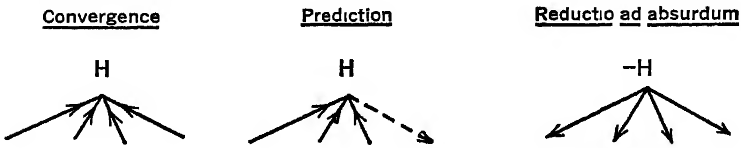
	hypothesis $\supset$ facts	$P \supset Q$
	facts untrue (because absurd)	$-Q$
$\therefore$	hypothesis false (and its contradictory true).	$\therefore\ -P$

This, of course, is a perfectly valid procedure. We recognize it as one of the correct forms of the implicative argument. May we not, then, assume that the *reductio ad absurdum* method gives proof rather than just probability?

Unfortunately it is not as easy as that. In the case of the "murder" on the bridge, the local constabulary *might* be able to show that the note held in the victim's left hand had some special significance. It might also be able to show that the governess was carrying a gun because of her fear of a savage mastiff at large on the grounds at night. And surely it could contend that the governess committed the crime impulsively, and was so shocked and bewildered by what she had done that she took thought neither to establish an alibi nor to get rid of the weapon. In brief, the word "absurd" is a dangerous word. It is used far too often emotionally and superficially, to dismiss an idea against which the mind is closed. When Sherlock Holmes dismissed an idea as absurd it *was* absurd. But he was a character in fiction. In the real world is there anything that is truly absurd?

The Pattern of Our Dealings with Hypotheses

These three methods of working with hypotheses are easily remembered because they form a triad with a pattern of which predication is the center:



Convergence is purely convergent, has all of its arrows pointing toward the hypothesis. Prediction is both convergent and divergent, with arrows pointing both toward and away from the

hypothesis. *Reductio ad absurdum* is purely divergent, with all of its arrows pointing away from the contradictory of the hypothesis in which we are interested.

One does well to bear in mind, and this is no more important to the detective than to any one else, that in establishing hypotheses we should *never* say, "If P is true, then Q is true," but rather, "If P is true, then *it is probable that* Q is true." It makes no difference whether P stands for fact and Q for hypothesis (as in convergence), or P for hypothesis and Q for fact (as in prediction and *reductio ad absurdum*). In connecting facts and hypotheses we can never rise above probabilities. Even when we employ the most careful scientific thinking and control our experiments with infinite pains we are still in the realm of the probable, as we shall discover in the following chapter.

Chapter Six

SWANS, SUPERSTITIONS AND THE SCIENTIFIC METHOD

GOOD logic may be a question of life or death. Many who read these words owe their lives to logical thinking. Anyone who has anxiously watched a doctor bend in perplexity over the bed of a desperately ill patient will know and remember his indebtedness to logic. What causes this increasing weakness and inability to take nourishment? How read the symptoms? Is it a poison? Is it a germ? Is it a mechanical difficulty? Shall we use the stomach pump, or medicine, or the surgeon's knife? Something must be done and done quickly if the patient is to live. It was not many more than fifty years ago that whole communities of children were wiped out by diphtheria. Parents had ten children in the hope that half of them would survive childhood. To-day many doctors can only read about diphtheria. The life expectancy of the new-born baby is many times what it used to be. Men who could think clearly about causes and effects in nature have done more to take fear and pain and sorrow out of human life than has ever before been done in a corresponding period in human history.

The scientific attitude toward nature was slow in developing, partly because the logic of science was not studied until late. In fact it was not until the nineteenth century that definite steps were taken toward an understanding of the structure of

scientific thought. Man finds himself in a natural environment and is highly dependent on that environment not only for his comfort, but also for his life. Primitive man must have thought fire a good thing to have around; it warmed him, cooked his food, and kept wild animals away at night. How keep a fire going? Mr. and Mrs. Cave Man must have gotten a bad scare when their friends took sick suddenly and died in great pain. How avoid premature death? In other words, how does nature work? What are the laws of nature? What causes this to happen or that not to happen? If we can only find the causes we shall be able to exert some control over our environment, live longer and in greater comfort. What causes in nature should be linked to what effects?

Obviously there is something of wider significance here than in the type of problem discussed in the preceding chapter. When a detective is investigating a murder, or you are wondering what desperate thing your daughter is about to do, or I am puzzled about those girls wearing lampshades, we are studying individual instances and dealing with the particular facts involved in a single situation. You might say that we each have on our hands an individual event and are trying to find its particular cause. No attempt to find a law of nature, to the effect let us say that when a ladder is raised to a window some one is going to elope. No attempt to make broad generalizations. In these cases the facts themselves are of primary importance, and it is the particular facts involved which determine the tree-like structure we have been studying. All structures will be convergent or divergent, to be sure, but each structure will be uniquely attached to its incident. The structure we put down for my problem differs significantly from the structure of yours. It is only in a broad sense, then, that we speak of the logic of convergence and divergence. And, as we have seen, it is interesting more for its limitations than anything else.

But when you start to investigate the laws of nature you are dealing in broad generalizations covering innumerable instances. This problem is more profound. There is a great difference between "Smith murdered Jones" and "A bullet through the heart will kill a man"; "Olga is going to elope to-night" and "In the spring a young man's fancy lightly turns to thoughts of love." All four, as we shall see, are hypotheses; but only two are generalizations. But the establishment of generalizations requires more knowledge of structure.

The Development of a Generalization: the Famous Swan Problem

The swan is as famous in discussions of induction as Socrates is in the field of deduction. You and I are walking along a country road in England. Let us suppose that we know little about birds and that we have never seen swans. We cross a stream and see on the water a white swan. We admire the creature and are curious to know whether or not all swans have that striking pure white color. But there is nothing in the single instance to tell us one way or another. We remember that some horses are white; but others are black, others brown, others dappled, and so on. To induce from the sight of one white horse the generalization that all horses are white would obviously be erroneous. The very meaning of the word, "generalization," should tell us that the single instance in itself gives no basis for so broad an induction. It is the "One swallow does not make a summer" principle. Of course we do often generalize on the basis of a single instance. One taste of a good chocolate layer cake is enough to suggest the generalization that the next mouthfuls of that cake will taste good. But in such cases we are basing our generalization as much on past experience with food as on this first taste. The owner of a Chevrolet who

generalizes from the single instance of his own car to the effect that Chevrolets are easy on oil, is doing the same thing. But you and I are supposed not to be able to draw on past experience with birds, and hence find ourselves unwilling to generalize about the color of swans. So we continue our walk.

A few minutes later the road turns and follows the stream upward and we are favored with the sight of another white swan. You are still unwilling to generalize about the color of swans, as unwilling as you would be to generalize that an entire pack of cards is made up of hearts after you had drawn out two hearts. It is suggested to you, perhaps, that swans *are* white, but you are hardly willing to make a statement to that effect. To put the situation in a language with which we are now familiar, the sight of a single white swan makes it neither probable nor improbable that all swans are white. But the sight of a *second* white swan makes it faintly probable that white is always their color. The element of the accidental enters in here. If you see a man come into a room, toss his hat halfway across it and land it neatly on the hat-tree, you are going to comfort yourself with the idea that it was an accident. If, then, the man retrieves his hat, walks back to the door and tosses it neatly onto its peg again, your respect for the man's skill increases. You have tried the trick too often yourself without success, and you remember that accidents of that sort occur very infrequently. Well, one white swan might be an accident of nature, like an albino cat, but the chances that the second swan is also an albino are smaller.

A week later you and I are walking in the city of London and in a public park we see a white swan paddling toward us. This time you will surely say to me: "You know, I am beginning to think that *all* swans are white." And as you continue to go around, seeing different swans in widely separate parts of the country, on different bodies of water, swans of different

ages and conditions—and they are all white—the probability is increased that every swan is white. If the man with the hat continues to throw his hat across the room and onto the hat-tree it becomes more and more probable that it is a matter of skill, but no matter how many times the trick was accomplished there would not be certainty.

In offering the "Rough or smooth?" alternative to my tennis opponents this spring I have lost consistently and without exception, but it would be foolish to attribute this to the skill of my opponents. Curious sequences of events do occur. Remember the days when "everything goes wrong"? And so with the swans. You and I are getting fairly cocky in our new-found knowledge of birds and you announce that you believe that all swans are white. It would be the sheerest accident if you traveled all over England and saw only white horses—though of course *it is possible*. It is said that when Catherine the Great traveled on the Volga whole companies of theatrical actors and managers went ahead of her and arranged all of the "signs of prosperity" and demonstrations on the part of the populace. The Soviets are accused of doing the same thing to-day. It may be that some one has gone around ahead of us "planting" white swans to fool us. That would be pretty silly. Still, there are silly people in this world.

It is a matter of history that some one did do just what I have pictured us as doing. He examined all of the swans he could find in England and came to the conclusion that every swan is white. It seemed fairly safe, in fact it seemed entirely safe. And people in England took his word in the matter for years. Then certain adventurers made the mistake of voyaging to Australia, and there they were completely taken aback to find *black* swans. Of course they had only to find *one* black swan to destroy the dictum about swans being white. That is the logician's law of gravity working again. It takes only one contra-

dictory example to destroy the most cherished and most carefully stated generalization.

When we build generalizations according to the method of *Simple Enumeration*, as this process is called, we are like children building a tower out of blocks. One block by itself could hardly be called a tower. Two blocks, one on top of the other, might in the childish imagination be so named. As additional blocks are added to the pile the structure looks more and more tower-like. The difference is that in building a tower the more blocks are added to the pile the easier it is to topple over the thing; while in building generalizations, the more the instances the more secure the generalization. The case of white *vs.* black swans is highly unusual in this respect. That is why it is quoted so often. But in either case it takes only one good blow to destroy the entire structure.

The Danger in Simple Enumeration: the Rise of Superstitions

While we are on the subject of black animals, consider the superstition about black cats. How do you suppose it arose? I imagine that Joe Yoekel was walking along the highway one fine morning when a black cat crossed in front of him. He took no particular notice of the incident. But a short while later he was set upon by thieves, beaten and robbed. When he was able to gather himself together and walk home he told Mrs. Yoekel the catastrophe (no pun intended), and added "I know that the desperadoes came out of the woods near Bill Blacksmyth's house because I remember that just before I came to his house a black cat crossed the road in front of me." That innocent remark is enough to start the ball rolling. Mrs. Yoekel says: "A *black* cat? That's funny! The day my poor old mother fell down the cellar stairs and broke her leg a black cat had just

jumped up on the window sill of the kitchen." At this point Mr. and Mrs. Yoekel probably regard the two appearances of black cats at these unfortunate moments as coincidences. They think no more of them.

But it just happens that Jerry Doolittle and his sister come to call in the evening to hear about the robbery. And after Joe has told his story Mrs. Yoekel is eager to share the honors of the conversation, as some women are, and she launches into the story of the black cats. It happens, shall we say, that Jerry's sister is an emotional nit-wit who loves the mysterious and lives on gossip. The story of the two black cats sets her thinking, if you can call it thinking. She remembers the black cat that inhabits the haunted house over in the next county. Or perhaps she was badly scratched by a cat (it doesn't have to be a black one, either) and has disliked cats ever since. Her tongue starts wagging, and before many moments the superstition about black cats and bad luck is well under way. Our four characters spend the evening trying to remember incidents in which black cats figured—and of course human psychology is such that they remember the *un*lucky ones. They remember that the day a black cat was seen in the squire's barn a farm hand was wounded painfully by a pitch-fork.

By this time our company is looking for trouble. They forget that on the same day the farm hand found a sack of money by the side of the road. You may be sure that by the end of the next day, thanks to Jerry's sister, this new idea about black cats is all over the village, and four hundred people instead of just four are connecting black cats with bad luck. It is a case of remembering what one wants to remember. The one or two rational members of the community who are honest enough to recall that sometimes the appearance of a black cat can be linked to a joyful event go completely unheeded.

Do you remember the time you first encountered a new word

or a new name? And do you remember that you kept meeting it in your reading afterward? The principle is the same. Some one tells you about the Count of Monte Cristo, and thereafter you keep seeing references to the man. The fact of the matter is that you encountered him as often before as after. But before you knew much about him he had no meaning for you and you forgot him quickly. Furthermore, the times when you encounter a new word or name and *do not* meet it again soon will not be remembered. Your memory is working in a highly selective manner, seeking uniformities, enumerating similar instances, until you are willing to believe that a new word will always be encountered again the same day. When it works you note the fact: when it does not the fact is not brought to your attention.

All of the old superstitions about dreams fit into this picture also. You come down to the breakfast table and report a dream about receiving a letter containing good news. In the morning's mail comes a letter from a law office notifying you that a distant relation has died and left you a sum of money. Do you remember the dream? Of course you do! But suppose that you dream of seeing an ambulance in front of the house, yet every one in the family remains in the best of health for weeks. Do you remember that dream? Of course not. There are thousands and thousands of dreams every time the sun goes down. Some of them are bound to coincide, if only remotely or symbolically, with events that occur when the sun rises again. Whenever they do coincide they bolster a superstition; whenever they do not they sink into oblivion.

Hence the method of Simple Enumeration is dangerous, not only because the single contrary incident can easily overthrow the generalization but also because human nature is such that we single out uniformities from the series of heterogeneous events and are willing to forget the contrary incidents. Human

nature does not change much, either. We are still inventing new superstitions. Most interesting is one which in its original setting had a true basis in fact, and hence is an aristocrat among superstitions.

Many otherwise intelligent people refuse to be the third to light a cigarette from one match. When taxed with the superstition they smile, but say it does no harm to be careful. Gossips tell us that this superstition was launched by manufacturers of matches to increase their business. Actually it arose among the soldiers in the Boer War. They observed that the third man to get a light from one match was apt to get hurt. And this was probably true. Out of that fact arose the superstition that the third on a match was unlucky. In the beginning there was a good reason why the third man got hurt. If at night a match burned long enough for three men to get a light from it, enemy sharpshooters were given time to aim at the flame and shoot, and the third man sometimes got wounded. It is well known that men who live daily in great danger become intimate with Lady Luck. In this case instead of scientifically seeking the cause of the phenomenon, they were willing to stop with the fact of the match and let the Lady do the rest.

It is a far cry from the front-line trenches to an afternoon tea, but in both situations people are lighting cigarettes with matches. Hence the superstition has been carried on long after the true cause was removed. The point here is that at first the enumeration of uniformities was reasonably correct; only after the war did people remember what they wanted to remember and forget what they wanted to forget. The fallacy was a *Post Hoc Ergo Propter Hoc* fallacy, "following after and therefore caused by . . ." This fallacy is a second major source of superstitions and has a part in all such emotionalized attitudes. The black cats appeared *before* the misfortunes and hence were thought to cause them.

Improvements on Simple Enumerations

It should be apparent that if we are going to make generalizations about phenomena in nature, about the color of birds or the causes of certain desirable or undesirable events, we have got to devise a more accurate structure for our thinking. John Stuart Mill developed what he called *the canons of induction* by way of defining accurate thinking about natural causes. There have been modifications of and improvements on his technique since his time. Every pioneer makes mistakes. We shall state and illustrate four modern methods of induction. It will be noticed immediately that they give definite procedures for the building of hypotheses about nature. The method of *Simple Enumeration* is too simple. It allows the weight of uniformities to operate, without attempting to control conditions or plan experiments in a rational manner. These four newer techniques give us a truly scientific method of increasing the probability of the hypothesis in which we are interested, make the investigation impersonal and objective, try to rule out coincidences and the effects of emotion.

The Joint Method of Agreement and Difference: Mushrooms and Black Cats Again

We are all to some extent scientific in our approach to the problem of cause and effect. Every time you eat mushrooms you are taken violently ill, let us say: when you omit the mushrooms you do not have trouble. Whenever you put water on a heated stove it will boil; but every time it is set down on a relatively cool surface it does not boil. Whenever there is a thunderstorm it is almost impossible to use the radio because of the static, yet after the storm has gone away the static has also gone. In other words when two events are found to be per-

sistently present together and absent together, it is probable that the first is the cause of or part of the cause of the second. Mushrooms cause illness in you: heat causes water to boil: thunderstorms cause static on the radio.

Employing this method is just a matter of noting presences and absences. There is no particular effort to control conditions. Using "C" for the cause in which we are interested and "E" for the effect, the situation which meets our requirements would look like this:

C, a, b	E, d, f
C, g, h	E, i, j
C, k, l	E, m, n
not-C, o, p	not-E, q, r
not-C, s, t	not-E, u, v
not-C, w, x	not-E, y, z

In noting the effect of mushrooms, for example, C stands for the eating of mushrooms, and E stands for the effect they have of making you ill, and small letters stand for irrelevant elements in the instances studied. "a" might stand for the fact that you also had steak at that meal, "g" for the fact that the mushrooms were creamed, "k" for the fact that you had ice cream for dessert the third time. "b" might mean that you ate at home, "h" that the next time you ate mushrooms at a restaurant, and "l" indicate that the third time it was maid's night out and your wife cooked the mushrooms. It becomes clear that where you ate, who cooked the meal, and what else you ate at the same time are not as important as the fact that you ate mushrooms. Similarly with the other left-hand factors, "o" to "x." They seem less important than the uniform fact that when you did *not* eat mushrooms you were *not* ill.

In dealing with problems of causal connection the dotted line indicates a lapse of time. The small letters on the right-hand side stand for factors in the effect that are irrelevant. "d"

might indicate that the first time you were so alarmed that you called a doctor, and "i" stand for the fact that the next time you tried home remedies. "f" might mean that the meal was very expensive, and "j" record that the second time there was no expense to you because some one else paid the bill. "m" may stand for the fact that you were taken ill while you were at the movies, "n" for the fact that the third time you were not actively and violently ill. And similarly with the right-hand factors from "q" to "z."

Our friend the black cat will not be treated so harshly when care is taken to employ this method. If it had been true that the appearance of a black cat was always followed by misfortune and that no-black-cat was followed by the absence of misfortune we should have a case against the animal:

black cat, a, b	. misfortune, d, f
black cat, g, h	. . . misfortune, i, j
no black cat, o, p	. . . no misfortune, q, r
no black cat, s, t	. . no misfortune, u, v

But our superstitious friends forget other instances that did not fit this picture. They forget that the time Blacksmyth burned himself so badly neither he nor his wife nor any one connected with him had seen a black cat for weeks.

no black cat, a, b	. misfortune, d, f

And they forget the day that the local harness-maker had been befriended by a family of black cats, had had three cross his path the same afternoon, and had been annoyed by a black cat caught on the roof of his house that evening, yet had had nothing of an unusual nature happen to him that day or in any day even remotely connected with these incidents:

black cat, g, h . .	no misfortune, i, j

The Society for the Prevention of Cruelty to Animals should surely promote the study of logic. Fewer people would carry

rabbits' hind legs in their pockets, and the cry of "Mad dog!" would be heard less in our cities on hot summer days.

Of course the harness-maker's experience offers a far more serious obstacle to the superstition about black cats than does Blacksmyth's. For any given effect there are many possible causes. It may be that Blacksmyth had walked under a ladder that morning, or it might be Friday the thirteenth, or perhaps he was born under an unlucky star. If superstition cannot find one cause for misfortune it will find another. The superstitious are a hardy race! But what about the harness-maker? Here were innumerable black cats which even with their combined strength could not dig up trouble. Superstition will find a way of getting around this, too, but it has greater difficulty.

In the case of the mushrooms the situation is somewhat more clear. There are a good many things that can cause me to be ill. Perhaps the meat has spoiled. I might be poisoned by the cook who has just been given notice. It might be that I had exercised violently too soon after eating. Any one of these causes would have produced the same effect. In our new Magic-of-Science shows we see that water can be made to boil without fire. There is generally a plurality of possible causes of a given event. We should not be alarmed to find instances in which the effect occurs without the cause which we suspect as most often producing it. But when I eat mushrooms and am *not* sick, if water does *not* boil when placed over a fire, it is a definite indication that I am on the wrong track in seeking to attach a cause to a given effect.

The Joint Method Has the Same Structure as Implication

If you are interested in making comparisons you will find a close and interesting relationship between the technique of the

method we have just described and the structure found in the study of the implicative argument. If you set the condition that *P implies Q*, you allow three possibilities:

1. P is true, and Q is true.
2. P is false, and Q is false.
3. P is false, and Q is true.

(1) If I eat too much ice cream, then I shall be sick. (2) If I do not eat too much ice cream, I may not be sick. And (3) If it is false that I eat too much ice cream, in other words if I do not eat too much ice cream, it is entirely possible that I shall be sick just the same, from some other cause. What I am saying is that too much ice cream *causes* me to be sick. What I actually do when I say: "If I eat too much ice cream, then I shall be sick," is to rule out the possibility that I eat too much ice cream and *not* be sick! And, in general, what I actually do when I assert that *P implies Q* is to rule out the possibility that P be true *and Q false.*

In a similar fashion, when I am trying to connect a cause with an effect by the *Joint Method*, I shall be encouraged by the discovery of three types of sequence:

C, a, b	E, d, f
not-C, g, h	. . not-E, i, j
not-C, k, l	E, m, n

The only sequence that will force me to abandon my investigation will be the one in which the "cause" appears *without* the effect:

C, o, p	*not-E*, q, r

Hence this method is somewhat more complex than its name indicates. We look for cause and effect appearing together, cause and effect absent together, *and* are not afraid to accept situations in which effect appears without cause. It might be simpler to

define this method negatively as one in which the cause *does not* appear without the effect: i.e.

It is not true that (C, a, b   not-E, d, f)

but we are apt to shy away from negative definitions as we do from negative commandments. Better to stick to the three-fold definition, which tells us positively what we may expect in a causal situation.

Further Improvements of Technique Are Needed

But if we are going to be truly scientific in our study of nature, the method we have just outlined, while an improvement over the process of simple enumeration, is insufficient. From the symbolism above it is evident that the *Joint Method* gives more attention to C and E than to the "irrelevant" factors. It makes no attempt to control these other factors, it is too naïve to provide a rigorous test. Truly scientific experimentation begins when we give more direct attention to the *other* factors in the case. We perform experiments in chemistry "under laboratory conditions": we employ "control" groups in psychology: in all experiments we try to vary as much as possible the presumably irrelevant factors so that we may know that they *are* irrelevant. We become scientific. This leads to two other types of thinking, each more accurate than the ones so far discussed. Some experimental situations suggest the use of one, some the use of the other.

The Method of Agreement: The Problem of the Yacht Designer

An amateur yacht designer is interested in demonstrating to his pals that a hull he has built is faster than the others at

the club. He wants to establish that it is the lines of his boat that cause it to win. In the arguments that arose at the club-house various other explanations were suggested. "It might be his peculiar skill; or the cut of the sail; or the weight of the crew; or the nature of the wind, e.g. its strength or direction. Many victories have been ascribed to one or other of these factors. How eliminate them all? How show that the style of the boat was alone relevant? It took time. He allowed each member of his crew to handle the boat for a race. She won every time. That eliminated the hypothesis that his own peculiar skill was the cause of the winning. He sailed in heavy winds, middle winds, and light winds. That eliminated strength of wind. He sailed in east winds, west winds, north winds, and so on, and won each time. That eliminated direction of wind. He borrowed another cut of sail, substituting gaff rig for his marconi rig, and won again. That eliminated sail cut. He varied the combined crew weight, substituting heavy men for light men, and won. That eliminated weight of crew. Within the limits of one season, he varied every factor he could think of, and won each time. The only factor common to all trials was the style of boat . . . the hypothesis was accepted as proved." [1] Had he actually proved his case?

Let us put the entire procedure into symbols and see how it looks. First of all, give the following letters these meanings:

D—design of hull	E—east wind
O—owner at tiller	S—south wind
F—friend at tiller	N—north wind
C—cousin at tiller	W—west wind
H—heavy wind	G—gaff rig
M—middle wind	R—marconi rig
L—light wind	A—heavy crew

B—light crew

[1] Castell: *A College Logic* (Macmillan), p 234-5.

V—victory
P—sailed on port tack at finish
St—sailed on starboard tack at finish
I—all boats finished
J—two boats disqualified
K—one boat dismasted
Q—time unusual
T—slow race
X—designer stayed at Club for dinner
Y—designer dined at home

The symbolic expression of the experiment performed by the designer would then look like this:

D, O, H, S, R, A	V, P, I, Q, X
D, F, M, S, R, A	V, St, J, T, Y
D, C, M, S, R, A	V, P, I, T, Y
D, O, H, E, R, A	V, St, I, Q, Y
D, F, L, E, R, A	V, St, I, T, X
D, O, H, N, R, A	V, P, I, T, Y
D, O, H, S, R, A	V, St, I, T, Y
D, O, L, E, R, A	V, St, J, T, X
D, F, L, W, R, A	V, P, I, T, Y
D, O, H, N, G, A	V, P, J, T, Y
D, O, M, S, G, B	V, P, I, T, Y
D, O, H, W, R, B	V, St, I, T, Y
D, O, H, W, G, A	V, P, K, Q, Y
D, C, M, W, R, B	V, St, I, T, X

The constant agreement of "D" and "V" in the diagram clearly suggests that the design of the boat was responsible for the fine string of victories. The design of the hull remained unchanged, and the performance of the boat was uniform. Because of this character of agreement, this method of establishing an hypothesis is called the *Method of Agreement.*

During more than half of the sailing season our friend might have thought that the rig of the boat was responsible for its achievements, for "R" was a constant factor in the first nine trials. But when the rig was changed and the boat continued to win, the designer was justified in supposing that it was not

the rig after all that made the boat so fast. Surely this picture of the data makes it probable that the design of the boat was responsible for the victories. We may imagine that the designer was able to put up a very strong case in the long winter evenings at the Club when they were hashing over the events of the past season. But if there was a logician in the crowd he could make an interesting analysis of the data.

He would notice, for example, that in two of the three trials in which a gaff rig was employed the wind was heavy; and that the one time when the gaff rig was used but the wind was only medium, the crew was unusually light. It is possible, then, that in most of the trials the marconi rig was the cause of the victories and that when the gaff rig was used the boat won by virtue either of a heavy wind that was favorable to that type of rig, or by virtue of a combination of middle wind and light crew.

He might also notice that the owner almost always sailed the boat in a heavy wind, the cousin in a middle wind, and the friend in light winds. This would suggest that the owner is a particularly good skipper on blustering days, that the cousin is good on average days, and that the friend is a shark in handling a sail boat when there is hardly a breath of air stirring. Of course, on the eighth trial the owner took the tiller when the wind was light, but perhaps on that particular day he had the advantage of a tidal current that did not favor the other boats because they stayed closer to shore. Also, on the second trial the friend won with a middle wind. But on that day two of the contestants were disqualified, and they may both have been at the time ahead of the boat in which we are interested.

The logician would also seek other factors which were uniform throughout the season. Had the designer eliminated all of the other possible causes of the boat's winning so consistently?

Can you think of other factors which were not taken into account? We do not know whether or not the owner was in the boat on every trial. If he were, it is entirely possible that he possessed a knowledge of tidal currents superior to that of the other contestants and that that knowledge was at the disposal of whoever took the tiller. Did he try shifting ballast? Perhaps the sails he used were newer than the sails on the other boats.

The most obvious constant factor, of course, is the hull of the boat itself. It is surely probable that something to do with the hull is the cause we are seeking. But need it be the design? Even discounting the possibility of a hidden electric motor, we cannot be sure that the *design* of the boat is the most important factor in the case. It may be that all of the boats but the one in question were moored in salt water and hence had marine growth on their bottoms which would slow them up, while the designer's boat was moored up river in fresh water and hence was relatively free from marine growth.

In short, there is nothing in the experiments of the designer which *proves* that his is a superior design. Too many other factors enter into the situation. None of them, to be sure, are as probable as the cause which he suggests. Hence he may rightly say that *it is probable* that the design of the boat *or something to do with the hull* is the cause of the long string of victories. But he must nevertheless realize that he is dealing in probabilities. The *Method of Agreement* is not infallible.

In a situation such as the one just outlined the probability that the true cause has been found will depend largely upon the willingness of the experimenter to vary the conditions not considered relevant. The designer varied winds, crews, rigs, skippers, etc. It is in this variation of irrelevant factors that the method he employed has its advantage. In the *Joint Method of Agreement and Difference* attention is paid exclusively to the effect and the suspected cause, to see whether or not they

are present together and absent together, and to insure that the cause is never present without the effect. In the *Method of Agreement* the presumably irrelevant factors are given careful consideration in the hope that one by one they may be eliminated. The *Method of Agreement* is correctly characterized as a method of elimination, working on the principle that the more factors are removed the more probable it becomes that certain remaining factors are the true cause and the true effect. How does the process of elimination work?

Different Kinds of Causes and to What Extent They Are Eliminated by the Method of Agreement

At this point we must make a distinction between *necessary* causes and *sufficient* causes. A necessary cause is one without which the effect could not take place. For example, oxygen is a necessary cause of the lighting of a match. Without oxygen the match could not be ignited. A sufficient cause is one sufficient to produce an effect. Obviously, oxygen is not sufficient to cause the lighting of a match. However, a bullet through the heart is sufficient to kill a man. Hence it is a sufficient cause of the death of a murder victim. But it is not a necessary one. There are many other more or less attractive causes of death. If we apply these distinctions to the boat designer's experiment we shall get interesting results.

The fact that the owner was skipper during some of the trials cannot be a *necessary* cause of the victories, for the boat won a number of times when one of the other two was in command. Heavy wind could not be a necessary cause, because the boat won in middle and light winds. The factors of the direction of the wind, of the nature of the rig, and of the weight of the crew are similarly eliminated as *necessary* causes.

Any one or combination of these factors may have been sufficient to cause victory at various times, but the probability that in this case the victories would be consistent is small. We should be faced with a large number of coincidences.

Our common sense and the obvious interest in victories makes it almost unnecessary to point out that certain factors in the effect are also eliminated. For example, the design of the boat could not be the cause of the fact that the designer stayed at the Club for dinner after the races: the design remained unchanged, yet the designer's meals were sometimes at the Club and sometimes at home.

We see, then, that this experiment has eliminated certain factors ineligible as necessary causes of the effect in question, and has also made the probability that they are sufficient causes smaller than the probability that the design of the boat is both necessary and sufficient to cause the victories. Everything points toward the design as the factor in which we are interested.

This *Method of Agreement* is effective but tricky. If a number of students at a dormitory have all had fish for supper, though the rest of their dinner was varied, and all are ill that night, we can with high probability eliminate the different desserts or drinks as the causes of the trouble. Everything points to the fish as the cause, and the fish dealer may lose a big customer. But the danger with this method is two-fold. Sometimes the factor which is uniform in each instance studied is not carefully enough analyzed. At others, there is some other uniform factor which has not been considered. Thus it may not have been the fish that caused the illnesses, it *may* have been the sauce on the fish! If so, the fish dealer is in danger of getting a raw deal in exchange for his raw fish. Or perhaps some poisonous ingredient had gotten into the drinking water in that dormitory and caused the illnesses.

In the days of sailing ships and long voyages it was known

that fresh vegetables prevented scurvy. Not understanding the more precise cause, sailing masters went to great trouble and expense to provide vegetables for the crew. Now we know that a vitamine ingredient of vegetables was the cause, and the more easily kept lime juice (which contains the necessary vitamines) has become part of the sailor's fare. The uniform factor has been more carefully analyzed. More recently, to choose another example, it has been argued that Latin should be studied in high school because those who study it do well in college. But there is another uniform factor in the high school situation which is not often enough taken into account. Those who take Latin in high school elect it because they are good students!

The Method of Difference: the Bacteria Problem

If you had been the dietician at the college where so many were taken ill you would not be satisfied to discover that all who had eaten fish had become ill. You would immediately inquire into what had happened to those who did *not* take fish. If you discover that you are getting uniformly poor results with your camera you will not blame the establishment that did the developing for you until you have tried *some other establishment.* In other words, it is not enough to find that whenever the cause which you suspect is present the effect in which you are interested occurs. You will immediately think to *remove* that cause and see what happens. This introduces the second carefully thought out method of connecting causes and effects, known as the *Method of Difference.* Wherever it can be applied it should be. But it is not always possible to use it. In the case of the winning sail boat it would be impossible to change the design of the hull and see what happened.

Our friend the designer could experiment *only* with the *Method of Agreement*. That is why we studied his case so carefully. But in most situations the Method of Difference is applicable. More than the others, it requires a laboratory technique, however.

There is the familiar experiment to "prove" that the atmosphere contains bacteria. You take two test tubes, each containing a culture favorable to the growth of bacteria, the *same* culture poured from the *same* flask. You sterilize *both* test tubes and their contents by placing them in the *same* electric oven for the *same* length of time. You then seal the first of the two with wax, and expose the second to the air for a period of time. After the exposure the second is also sealed with wax from the *same* source and with the *same* instruments. The two sealed test tubes are then placed in the *same* temperature, favorable to the development of bacteria, left there for the *same* period of time. Finally the two test tubes are unsealed with the *same* instruments and samples of each culture studied under the *same* microscope. The first is found to be free of bacteria, while the second shows a thriving colony of germs:

Notice the frequency with which the word, "same," appears in this description. It gives the key to the entire procedure. If we employ the following letters with these assigned meanings:

A—exposed to atmosphere for a given period of time.
T—test-tube container employed.
C—culture used.
S—process of sterilization.
W—wax used for sealing.
I—instrument employed for sealing.
D—temperature used to encourage development of germs.
P—period of time at this temperature.
U—instrument used for unsealing.
M—microscope used for examination.

G—presence of germs at conclusion of experiment.
C—color of culture at end of experiment.
E—experimenter making the final test.
X—condition of wax at end of experiment.

we may symbolize the experiment in the following manner:

A, T, C, S, W, I, D, P, U, M	G, L, E, X
Not-A, T, C, S, W, I, D, P, U, M	Not-G, L, E, X

How adequate is this as a proof that there are germs in the atmosphere?

If two experiments are exactly alike except for one detail, and the results are the same except for one feature, we are surely led to suppose that the one difference in the original conditions is responsible for the one difference in the effect. You and I eat the same things for supper with one exception, and do the same things before and afterward; then, if you are ill that night and I am not, the general supposition would be that the thing you ate which I did not was the cause of your illness. But we cannot be certain of this. It may be that you have been taken with appendicitis; it may be that the movie we saw had an unusually bad effect on you because of the horror scenes; it may even be that some one slipped a poison into the food on your plate. In other words, you could not with absolute assurance blame your food for your trouble.

And the same thing is true even of a laboratory experiment. No two experimental situations can be alike in every detail and respect. The presence of the bacteria in the second test tube *might* be caused by the fact that this test tube was not free of germs at the beginning of the experiment and these germs had not been killed by the oven heat. More probable is the possibility that in sealing the second tube not enough care was taken to keep out germs. The second wax seal *might* have germs on it. Also, during the final process of examining

the cultures under the microscope something unobserved *might* have happened which would place germs on the second slide. The good experimenter will, of course, repeat the experiment several times, and if the results are always the same, the chances of any one of these other causes being responsible are appreciably reduced. It is improbable, though not impossible, that the same accident (or even different accidents leading to the same effect) will occur more than once or twice.

This *Method of Difference* is also a method of elimination. It does not show conclusively that the cause in which we are interested is the one that produced the effect, but it does show that certain elements in the causal situation are irrelevant to the effect. For example, the nature of the solution used as a culture is eliminated as a *sufficient* cause of the presence of germs, for in one case the germs did not appear. Similarly, the temperature treatment of the test tubes cannot be a *sufficient* cause of the result in which we are interested, for in the first case there were no germs. If the culture for the first test tube had been poured from one bottle and the culture for the second poured from another, the experiment would have been far less valuable. And if different temperature treatments had been employed with each container the results would have been far less significant. The reader will quickly figure out for himself which factors in the *effect* are eliminated as not necessarily produced by the causal factor of exposure to the atmosphere.

The danger in using the *Method of Difference* will be either that there is an undetected difference in the causal situation, or that the difference which is taken to be the true cause of the dissimilarity in the effects is not sufficiently analyzed. The former possibility has been illustrated in the preceding paragraph. What about the latter? The difference in the treatment accorded the two test tubes is that only one was exposed to

the atmosphere for a period of time. But still the germs may not have come from the atmosphere. Can we be sure that while our back was turned a fly did not buzz into the open test tube and wipe its six dirty feet on the glass? Or may it not be that *the moisture* in the air added a condition particularly favorable to bacteria? Or some chemical constituent of the atmosphere?

One must be very careful. For a long time it was thought that darkness frightened children, for in the dark they were afraid and out of it they were not. It was not until recently that it was discovered that it was always something that happened *in the dark*, usually a loud noise. Take a lot of a certain patent medicine and you will cure a cold quickly: do not take it and the cold hangs on. But still the virtue may not be in the medicine itself, it may simply be that with the medicine you are drinking a lot of water.

One of the difficult things about raising a child, as every parent knows, is the fact that you cannot use this *Method of Difference*. To spank or not to spank, that is the question. But, alas, you cannot spank soundly and study the results, then not spank and look for the difference. When you try not-spanking you are trying it on a child that has already been spanked. And when you spank a child on whom forebearance has been tried you are up against the same difficulty. An unavoidable difference has entered into the situation. Shall I send my boy to Harvard or Yale? You cannot try them both! It is extremely difficult to experiment with human beings. We have lately developed the control-group technique, but even that has its difficulties. Suppose we want to "prove" that the direct method of teaching a language is more successful than the older method of studying from grammar texts. We pick two classes as close in achievement and average I.Q. as possible, teach one by the direct method and the other by the

older method. The results are decidedly in favor of the direct method. But if different teachers are employed we cannot be sure that one teacher is not far better than the other. And if the same teacher handles both groups she will surely have more interest in one method, or be better adapted to one method, and hence teach one of the two better.

Even in the laboratory, *two* experiments must always be performed. And we cannot be sure that conditions in the two are absolutely identical. If one person performs both experiments we shall have a uniformity of technique but a different time factor: if two persons perform the experiments the techniques of the two may make quite a difference. In studying bacteria we had to use *two* test tubes. Shall we demand that they be made by the same firm, from the same ladle of molten glass? That would obviously be going too far. But we must insist that they be washed at the same time, in the same water, with the same thoroughness, etc. How far should we go in making our demands? And think of what might happen to the instrument used for sealing on the wax, between the first sealing and the second. One accidental touch of an unsterilized object and the whole experiment is void. Yet it would be perilous to use different instruments, too. The best we can do is to repeat the experiment often enough so that the probability that accidental variations have determined the difference in the results is decreased to a minimum. Even then the experiment is not a "proof."

The Method of Agreement and the Method of Difference

What do we gain by repeating the experiment several times? In doing so we introduce the technique of the *Method of Agreement!* Suppose we perform the same experiment on

germs a week later, using a different pair of test tubes, another culture, a different process of sterilization, sealing wax for sealing, different instruments, different temperatures, etc. If we combine the results of the two experiments symbolically, we shall have the following:

A, T, C, S, W, I, D, P, U, M	G, L, E, X	(1)
A, t, c, s, w, i, d, p, u, m	G, l, e, x	(2)
Not-A, T, C, S, W, I, D, P, U, M	Not-G, L, E, X	(3)
Not-A, t, c, s, w, i, d, p, u, m	Not-G, l, e, x	(4)

This is the best scientific technique we have. It combines the advantages of both the *Method of Agreement* and the *Method of Difference.*

We saw that the former was a method that eliminated certain factors because they could not be *necessary* causes of the result in which we were interested. And we observed that the latter was a method that eliminated certain factors because they could not be *sufficient* causes of the result in which we were interested. Now we can eliminate irrelevant factors on the ground that they are neither necessary nor sufficient to produce the effect desired. For example, we observe that neither the culture *C* nor the culture *c* is *necessary* to the presence of germs, because we get germs without *C* (2) and we get them without *c* (1). And *C* is not *sufficient* to cause the result because when we use *C*, sometimes the germs appear (1) and sometimes they do not (3). Ditto with *c*. It is not difficult to consider in turn each of the irrelevant factors on the left-hand side and see why this experiment makes it so. Irrelevant factors in the result can be similarly treated.

But does this not take us right back to where we started? Are we not now employing the *Joint Method of Agreement and Difference?* No, we are not. A careful comparison of the symbolic structure of the two procedures will show this clearly.

When we employ the *Method of Agreement* and the *Method of Difference* we are controlling certain factors in the hope of being able to eliminate them as either insufficient or unnecessary to cause the result being studied. When we employ the *Joint Method* we are noting only the appearance of the hypothetical cause in its relation to the appearance of the effect desired, in the hope of establishing a positive relationship. Thus we may say that the former is a method of eliminating certain factors by careful experimentation, while the latter is a method of strengthening what seems to be a causal connection:

Agreement and Difference: C, ~~a~~, ~~b~~, ~~d~~, ~~f~~------E, ~~g~~, ~~h~~, ~~i~~, ~~j~~

Joint Method: C, a, b, d, f------E, g, h, i, j

The former is a negative method, the latter is a positive method. The former involves intelligent manipulation of the conditions of the experiment, the latter is the passive observance of the presence or absence of interesting factors in cause and in effect.

The Method of Concomitant Variations: The Problem of the Farmer

If you are trying to undo a particularly bad tangle in rope you often find yourself giving a tug on a certain loop to find out whether or not it is connected closely with another in the knot. If, when you tug, there is a corresponding movement in the second loop, you have established a connection. When you were a child you may have been puzzled at the movements of the counter-weight in an elevator shaft. What made it move? It was not long before you discovered that every time the weight moved up the elevator moved down, and *vice versa;* and that every time the weight stopped the elevator stopped. The two acted together. This suggests another method of

establishing connections in nature, one which is valuable in giving evidence in addition to that given by the methods we have just analyzed.

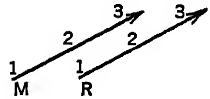
A farmer may observe that when he puts manure on his fields he gets a better crop. By the *Method of Agreement* he develops the hypothesis that manure causes good crops. If he is an intelligent farmer, scientifically inclined, he will now employ the *Method of Difference.* He will take two fields, as nearly alike as possible, plant the same seed at the same time, cultivate the two plots with the same frequency, but use manure on only one of them. If he gets better crops where he puts manure he has assuredly strengthened his hypothesis. But he has not yet exhausted the possibilities of experimentation. Suppose he takes three fields as much alike as possible, plants the same seed and farms them in the same manner, but varies the amount of fertilizer on each field. He puts a little manure on the first, a fair amount on the second, and a lot on the third. If, then, the crops on the first field are a little better than ordinary, the crops on the second definitely better, and the crops on the third much better, his hypothesis will be even stronger. Again giving letters meanings:

M_1—a little manure
M_2—more manure
M_3—a lot of manure
P—potatoes planted
S—nature of the soil
F—rainfall
C—frequency of cultivation

R_1—slightly better potatoes
R_2—definitely better potatoes
R_3—much better potatoes
B—potato bugs unusually bad
E—early harvest
X—market price of potatoes high

we can work out the following structure for the experiment:

M_1, P, S, F, C ——————→ R_1, B, E, X
M_2, P, S, F, C ——————→ R_2, B, E, X
M_3, P, S, F, C ——————→ R_3, B, E, X

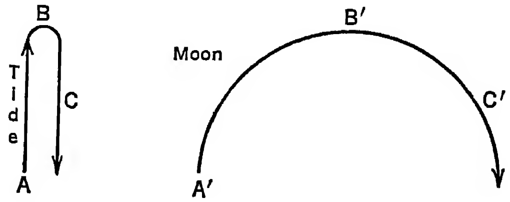


In many ways this is very convincing. The seed and the soil and the rainfall and the cultivation are sufficiently uniform that only through a highly improbable series of coincidences would they cause so definite a variation in the quality of the crop.

Accidental variations in soil and drainage might produce this result, though. The real danger, with this method, is that there may be some factor *other than* the one noted which is responsible for these constant variations. In this case, for example, the farmer might have become increasingly adept in sowing his seed, or the fields might be one above the other on a hill such that in a dry summer the middle field got more water than the top one and the bottom field got most of all. And there is always the danger that the causal factor being studied is not sufficiently analyzed. Perhaps it was not the manure as such that did the trick; perhaps it only served as a protection against the sun on a few hot but crucial days. Or perhaps it brought worms out of the ground and thus served indirectly to break up the soil. We are still dealing in probabilities.

There are cases in which this method, called the *Method of Concomitant Variations,* is the only one that can be used. What causes the rising and falling of the tides? This problem is an old chestnut among logicians. Experimentation in astronomy is limited. We cannot push the celestial bodies around at will, and we cannot control conditions the way we do in the laboratory. For example, you cannot try things with a moon and then without a moon. All you can do is to stand by and watch. But what you see can be eloquent. When the moon appears over the horizon the tides are at dead low. As the moon

ascends the heavens the tide rises. And when the moon reaches its zenith the tide is full. Then the tide goes down and the moon descends. Nothing could be plainer:



Also the time of high tide moves backward almost an hour each day, and so does the time at which the moon reaches its zenith. This method is closely kin to that of *Difference*. The irrelevant factors are constant, but instead of putting in and removing the hypothetical cause we now study variations in that cause, and instead of looking for presence or absence of the desired effect we look for variations in it. We then ask if these two variations seem interconnected.

There are situations, of course, in which the *Method of Concomitant Variations* is singularly inappropriate. You would probably not care to let one person eat a little questionable fish, a second person eat a full helping of it, and you yourself take two helpings; and then wait around to see how sick you all became! But in designing a boat you might be able to experiment with different angles of entrance of the hull and study relative performances. And in seeking bacteria in the air you might expose a series of test tubes to varying periods of contact with the atmosphere and count the number of bacteria developed per unit area on the slide of the microscope. On an unfamiliar radio you would find the volume control by experimenting with the knobs and finding the one whose turning varied with the volume. How about an experiment which might show a causal connection between wealth and

conservatism in politics? Or one that would show that quickness of reaction time makes the good baseball player? We employ this method all of the time, in stepping on the accelerator of a car, in determining the effects of alcohol, in connecting sun with sunburn.

Summary

These are the various ways of establishing the probabilities of generalizations about nature. *Simple Enumeration* is fair as a starter, but extremely dangerous because of the willingness of human beings to believe what they want to believe. It leads easily to superstition. One step better is the *Joint Method of Agreement and Difference*. There is still no attempt to control the conditions of the experiment, but we have advanced when we notice that not only is the effect present when the cause is there, but that the effect is absent when the cause is absent, and that the effect is never absent when the cause is present.

But our methods of studying nature become really scientific when we experiment, when we rationally and intelligently rule out factors as a way of strengthening a hypothetical cause. The positive methods, *Simple Enumeration* and the *Joint Method*, put a sword in the hand and an idea in the head. The negative methods, the methods of elimination, use the sword to cut down the false pretenders and to eliminate from the mind the irrelevant factors. The *Method of Agreement* eliminates necessary causes: the *Method of Difference* eliminates sufficient causes: the two together, when it is possible to apply both, eliminate both types of irrelevant cause. And the *Method of Concomitant Variations*, while psychologically most persuasive of them all, is only as powerful as the *Method of Difference*, indeed is simply a variation on that method.

When searching for hypotheses the methods of *Simple Enumeration* and the *Joint Method* will be most helpful, but when you are testing hypotheses you will need to apply the methods of elimination.

Chapter Seven

THE SCIENTIST AND THE PICTURE PUZZLE

WHEN you go to bed at night you may often wish that the sun would be a little late in rising the next morning, but you entertain not the slightest expectation that it will obey your wishes. The sun will rise to-morrow morning, and it will rise on schedule. You are as sure of this as of any judgment you make. The sun has always risen in the morning, and it will continue to do so. This would be a strange world if we could not count on the sun.

When I telephone my fuel dealer and order furnace oil I have the same confidence in the future. To my experience and the experience of others this type of oil has always burned when it came in contact with an electric spark. It would be a joke on me if next fall I should find that the laws of nature had changed and that the oil I had ordered (and paid for, presumably) was not combustible. Oil has always burned, and will continue to do so. If I were not sure of this I should sample whatever goes into the tank, and not pay my heating bill until the oil had served its purpose. But in my relationship to nature I do not act with so much scepticism. Why?

You and I have great confidence in nature. We assume that it will act in the future the way it has in the past, that it will obey certain "laws" which we have discovered. One of the most important things about our relationship to nature is that we

are not interested in what happened yesterday just for the sake of knowing what happened yesterday, of describing the past. We want to know what will happen *to-morrow*. Our interest in nature looks to the future. The farmer is not interested in what was responsible for his good crops last year just because of an inborn curiosity. Men are curious, but not that curious. The farmer wants to know what to do so that he will have good crops *next* year. The dietitian who is trying to find out what made the students in the dormitory sick is not just seeking a wicked revenge on the fish dealer, she wants to know how to protect the students from being sick *again*. The experiment seeking bacteria in the atmosphere would be of little interest if the air were not going to *continue* under the same conditions to contain germs. The boat designer is concerned in knowing what makes boats win races, because there will be races *another* season. In studying nature our interest in the past looks forward to what will happen in the future.

The Sun also Rises To-morrow

Let us analyze the situation. How do you know that the sun will rise to-morrow? How do I know that the fuel oil I buy this summer will burn next fall? Your answer will be, of course, that the sun has *always* risen in the morning. There is no case in the memory of man when the sun has failed to rise. And my answer will be that fuel oil has *always* burned. In other words we are both counting on an observed uniformity in the past as being able to tell us about the future. Where do we get this idea that the past legislates for the future? Put symbolically your statement of the situation about sunrises looks like this:

Past Present Future

.......Sr Sr Sr Sr Sr Sr Sr Sr Sr Sr Sr (Sr) (Sr) (Sr) (Sr) (Sr) (Sr) ......

How do you get from the past to the future? What is there in the bald fact that the sun rose yesterday and the day before and the day before that and the day before the days before that, and so on, which tells us *anything* about *to-morrow* morning? The fact is that there is nothing. You go to bed serenely to-night because you assume that nature acts uniformly. You and your ancestors have observed that nature has acted uniformly up to this point, and you jump to the conclusion that it will continue to do so. You do not make a mistake in this, of course. The assumption has shown itself to be a very practical one. *But it is still an assumption.*

When you come to think about it, there is no particular reason to believe that it is true. Man has seen so little of nature: Someone has said that if you let the height of the Egyptian obelisk in Central Park, New York, stand for the estimated history of our planet, then the history of man would be represented by the thickness of a postage stamp stuck on top of it. Others have pointed out that if you represent the history of man by the twelve hours on the face of the clock and assume that it starts at midnight, then the *recorded* history of man did not begin until half-past eleven and man did not seriously study his environment until a few minutes before noon! And the history of our planet, to say nothing of the history of man, is just a drop in the bucket to the history of the universe! When we get over the headache of trying to imagine these time scales we see that human beings have studied nature during but an infinitesimal part of its existence. There seems to be a uniformity over that short period about which we can think intelligently, but we shall never know

whether or not nature is actually uniform. Indeed the chances are that it is not. But for all practical purposes we can assume uniformity, and do, and we base our study of nature on that assumption. If we did not take this step there could be no point in studying the past or experimenting in the present. Why study what happened yesterday and what is happening to-day if what takes place to-morrow is going to be entirely different?

Why We Assume Causal Connection in Nature

A professor of philosophy used to ask us what thoughts would cross our minds if the chandelier in the center of the room should suddenly expand like a balloon. To put the same problem more dramatically, suppose you and I were walking in the woods and came upon a stream that seemed to be running uphill rather than down. Notice that "seemed to be." "Illusion," would be our first reaction. We should rub our eyes and look again. If the brook still seemed to be misbehaving we should pinch ourselves smartly. Not awakening in our beds, we should commence a careful objective study of the extraordinary situation. All of these actions involve the assumption of a cause for the experience. Is it in our eyes? In the atmospheric conditions? Are we dreaming? If the brook is really running uphill, if the chandelier is really expanding, the next question is "Why?".

Like the men in Francis Bacon's *New Atlantis* we seek "the knowledge of causes, and secret motions of things; and the enlargement of the bounds of human empire, to the effecting of all things possible." *To the effecting of all things possible*, that is the point. We want to control nature. But where do we get the idea of causal connection? Fire causes water to boil? Fertilizer *causes* good crops? Bad fish *causes* ptomaine poison-

ing? No one ever saw, smelt, tasted, heard or felt a cause. Where do we get the idea? Everyone uses it but no one knows much about it.

It is demonstrable that wherever we "find" causal connections, the only elements which we actually experience are the *contiguity* of two events and the *priority* of one of them. Fire and boiling water are contiguous. The strewing of manure comes before the good crop of potatoes. If we were a species interested in the past for its own sake, we might have spoken of a relation of "priguity" or "contigority," and dismissed it as unimportant. We might have been a race of mystics interested not in controlling nature but in excelling in another, say the moral, sphere. But we *are* interested in causes. One of the cleverest stunts in advertising is to hang in a store window a coffeepot which shall seem to pour coffee endlessly and without replenishment. People gather around. But they have too much common sense to marvel at a miracle. They are trying to discover how the coffee gets back into the pot. What is the *cause?*

That events are connected in nature by a cause-and-effect series is another of our assumptions. Like the assumption of uniformity, it must always be an assumption; we can never know whether or not it is true. But also like the other, it is practical. *It works.* We want good crops. What are the necessary conditions of a rich harvest? We want to avoid illness. What is sufficient to cause ptomaine poisoning? You will surely have noticed that in all of the experiments diagrammed in the preceding chapter, only one factor to the right of the dotted line was significant. That was the element in the effect *in which we had an interest.* You notice, let us say, that every Sunday afternoon you feel lazy. Why? You disregard the fact that one Sunday afternoon you sleep, the next you just "sit around" and the next you read the paper all afternoon. You also dis-

regard the fact that one Sunday is in March and the next two in April. It is the laziness that interests you:

lazy, (sleep), (March)
lazy, (sit around), (April)
lazy, (read paper), (April)

What element in the past has been as uniform as your laziness? Or perhaps your tennis stroke works one minute and does not the next. Again, why? This time you forget about the uniform factors in the effect:

good stroke, (spin on ball), (opponent makes return)
bad stroke, (spin on ball), (opponent makes return)

What element in the cause has changed? Stance? Grip? Twist of wrist? We find that it is profitable to seek causes, so we continue to assume causal connections.

Nature Has Its Limitations

If we are playing the good old cut-throat game of Hearts, every trick on which the deadly queen of spades does not appear increases the probability that it will be thrown onto the next. Tension increases as the hand proceeds, until some poor unfortunate is caught. We realize that there are only fifty-two cards to play, and as each trick is taken there are fewer and fewer cards left. If there were a thousand cards in a deck the elimination of ten or twenty would make little difference. If there were an infinite number of cards in a pack, we might eliminate cards indefinitely and not know when a queen of spades would appear. What we have called the *methods of elimination* operate on a similar principle.

If you try to start your car in the morning and find that it will not run, you first look at the gasoline gauge to see whether or not it is empty. Then you put a metal screw driver

across a spark plug to see if the ignition system is working. Then you open the carburetor to find out whether or not the gas is coming through. There is a definite list of causes of failure in the gasoline engine. By the method of elimination you approach more and more closely to the true cause of the failure by eliminating others as irrelevant. You do this confidently because you assume that there is a limited number of things that can go wrong with the engine. This is easily illustrated symbolically.

Suppose we list the possible causes of the failure of the engine as ten in number:

A, B, C, D, E, F, G, H, I, J car won't run

When you are first sitting at the wheel stepping futilely on the self-starter, your heart in your shoes, the chances are *one in ten* that dirty distributor points is the cause. When you get out and start to fool around under the hood let us say G, H, I, and J are eliminated and only six possibilities are left. The chances are now *one in six* that filing the points will help. You have narrowed down the possibilities, and increased the probability that a cause not yet examined is the culprit. But suppose the list of possible causes were infinite. Pity the poor garage man: he would no longer be a saint in disguise:

A, B, C, D, E ∞ car won't run

It is characteristic of an infinite series that no matter how far you go in it you still have just as far to go to reach the end. The garage man could make a thousand tests on your erring car, and he would still have an infinite number ahead of him. He would be making no real progress, and if he did find the cause you could only describe him as extraordinarily lucky.

If there is an infinite number of mosquitoes in your bedroom it will do you no good to get up and kill twenty thousand: there will be just as many left. If you know the book

you are seeking is one of ten on a table, your chances of finding it in the dark are one in ten. If you know only that it is in one of several bookcases your chances of picking it out will be much smaller. And your chances of walking into the Library of Congress some dark night and getting it would be infinitesimal.

So in attributing significance to the methods of elimination we assume that the possible causes of a given event are limited in number. Logicians call this the assumption of Limited Variety in nature. Without it scientific experimentation would be absolutely pointless, for no matter how many factors had been eliminated as irrelevant we should have made no progress because there would be just as many left to consider as there were when we started. If we can suppose that something we ate for supper made us sick in the night our field for inquiry is small. If anything eaten during the last twenty-four hours might be responsible our problem is more difficult. But suppose the doctor looked troubled and wanted to know all of the things you had eaten during the last year, or during your life-time. Or suppose he wanted to find out what your ancestors had eaten that might have caused this trouble! The wider the field of possibilities the more useless does it become to say: "Well, it couldn't have been the chocolate pudding. My roommate had a helping of that, too, and he didn't get sick." This assumption also works. We shall never know that nature is not infinitely complex. But we act as if its varieties are limited—and get results.

The Argument from Analogy in Nature and in Man: Ostriches and Automobiles

We have seen, in the case of Denying the Consequent, that an argument dangerous in the field of deduction may be profit-

able in studying nature. This assumption of the limited variety of nature suggests another Dr. Jekyll and Mr. Hyde among logical structures. We were warned that in an argument the employment of an analogy is usually highly specious. Seldom is the analogy valid. But, curiously enough, the argument from analogy is really helpful in studying nature.

If I attempt to argue that a human being is like a machine and that therefore, like a machine, he is not responsible for his actions, there are many who would question the adequacy of my argument. But if I am walking along a desert waste and see two ostriches with their heads in the sand, an argument by analogy will be in order. I examine what I can see of the two birds. They have the same size bodies, the same color feathers in the same pattern, their legs look alike, and their feet have the same number and type of toes. Every comparison I make of the two shows them to be alike. It is then reasonable for me to argue that when they get over their fright and take their heads out of the sand, their heads will look alike.

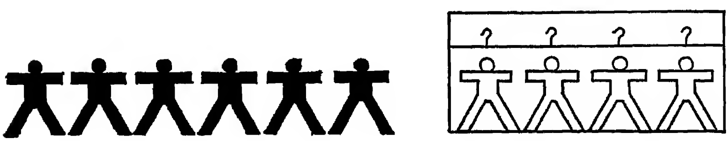
This would be a strange world if living creatures were so different from one another that they could not be classified. Cats are cats and dogs are dogs. Nature moves in channels. She develops what we call species. Suppose every "cat" were as different from the next as it is from a dog or a horse! One of the wonders of nature, as every parent knows, is that most children are born with two arms and two legs, and a head with the usual number of eyes, noses, ears, mouths and chins. Of course the number of chins varies later on, but that is man's work, not nature's. Whenever a child is born with one eye, or a calf with two heads, that unfortunate individual is so rare as to be labeled a monstrosity and hired out to the sideshow of a circus.

Man also moves in channels. Houses built in different parts of the world, uninfluenced by one another, will have many

features in common. If you observed that two parked cars had the same name on the hub-cap, the same color body, the same wheelbase, the same design of radiator cap, the same number of doors, the same design of handle, the same number of ventilator slots in the hood, and so on, each item of similarity would increase the probability that if you lifted the hood you would find the same type of engine inside. Still, you might be mistaken. Some manufacturers put different engines in similar bodies. You have a probability, not a proof. If the two cars are parked on the same country road at the same time of night, have been parked the same length of time, and are similarly empty in the front seat, it is probable that the same thing is going on in back in both cases. But maybe not. Man is not so predictable as nature. Two men may be so much alike physically that they are identical twins, yet one may possess an appendix and the other not!

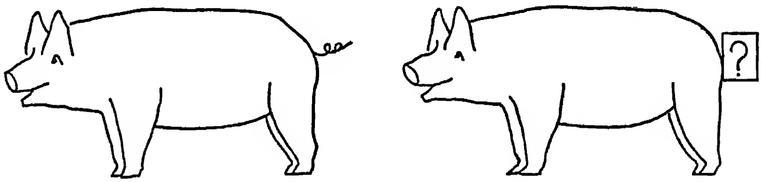
The Argument from Analogy and the Inductive Methods Compared

The difference between this experimental method and those described in the preceding chapter is striking. The inductive methods study two or more individual cases and proceed to induce from them generalizations which will cover individuals not yet studied:



The argument by analogy deals with only two individuals and proceeds to predict, from a study of a number of characteristics

which these two have in common, that they will have other as yet unstudied characteristics in common:



If I wanted to make a generalization about the shape of pigs' tails I should examine a number of pigs, find that all of their tails are curled, and assert it as probable that the next pig I see will have a curled cordal appendage. If I want to find out whether or not *this* pig has such a tail, knowing that another with which I am more familiar has one, I might start at the snout of each, listing the number of points in which the two animals are similar, and then assert that since there are so many similarities in the physical structure of the two it is entirely probable that they are also similar as regards shape of tail.

This may sound frivolous, but there are occasions when the method of analogy is the only possible one. We often wonder whether or not Mars is inhabited. At present there is no service, regular or irregular, which will transport us to the surface of Mars so that we can find out for certain. And our best telescopes do not bring us near enough to determine the question. But we do employ analogy. Did you ever ask yourself why it is *Mars* that interests us so much in this connection? It is because Mars is so similar to the earth in major respects. It has a similar history, a comparable temperature, an atmosphere, is subject to similar solar seasons; it revolves around the sun, it gets light from the sun, it is subject to the law of gravitation, etc. May it not also be similar in respect

to harboring life? If Mars were without water, like the moon, or experienced great extremes of temperature, like Mercury, we should not be so much interested in it. But we entertain the idea because Mars is so similar in many respects to the earth *and life has evolved on the earth!*

Another interesting use of this method is that of a military school in England which every year sends students to the battlefields of our Civil War to study tactics. The claim is that all of the major techniques of modern warfare were present in that conflict. Unfortunately it is true at the moment that future generals can easily go out and study a war in operation, but in our more civilized moments this is not possible and laboratory work in military operations has to proceed by way of analogy. The Civil War and modern war are so much alike in their fundamentals that when a new problem of strategy arises it can usually be studied at Gettysburg! If this seems fantastic to you, ask a student of military tactics.

The reader who is interested may easily discover for himself that most cases in which the method of analogy is employed can also be handled by the method of convergence. However, even though Analogy depends upon the assumption of Limited Variety, it is more logical in that its structure is more definite. Analogy is usually employed, as we have seen, when the facts in which the experimenter is particularly interested are inaccessible. The direct approach being blocked, he gets at them from behind. We do not know much about conditions a mile below the surface of the ocean, but we develop some highly plausible hypotheses by way of analogy with what we do know about lesser depths. No one knows anything about our Unknown Soldier except that he wore an American uniform and was found on an American battlefield, but that is enough to warrant a highly cherished analogy. Every time you buy a can of food you do so confidently because of the similarity of

that can and its label to those of others you have tested. Sometimes you are fooled, of course. The method of analogy is no more sure-fire than the others. But it does offer more rational grounds for argument than the ordinary convergence.

Nature and Jig-saw Puzzles

Our dependence on the basic assumptions employed in studying nature is not often realized. They serve as ways of connecting the individual bits of data which we gather. Just as we need mortar to hold bricks together in building a house, so we need assumptions to hold the individual data together in building probabilities. If we did not assume that nature obeys certain basic laws, each event in nature would be completely isolated from the others, and no one would have meaning beyond itself. People who have lived through war or revolution know the terror of the lawless and unpredictable. How terrible nature would be if one day fire boiled water and the next day froze it solid, if men died (or were born!) without cause, if each seed brought forth a unique flower or vegetable! The three assumptions we have outlined say the same thing in different ways: the events of nature are interconnected. They are connected in time from past to future (Uniformity), they are connected in their influences (Cause and Effect), and they are connected in that there are certain general trends (Limited Variety). These assumptions give us a framework into which we can fit facts.

Investigating nature is like putting together a picture puzzle. If we had in front of us a thousand pieces, each from a different puzzle, no one piece would fit with any other. But when we start on a puzzle we assume that there are definite connections between the pieces. The manufacturer of the puzzle might have played a low ungentlemanly trick on us. We hope

not. We suppose that the pieces belong to the same puzzle, are parts of a single picture. We assume that they are to this extent *uniform*. We assume that the pieces are also connected in the sense that if one piece has a curious loop, we shall be able to find another that fits it. We take for granted a sort of *causal* connection between the pieces. And we assume that the puzzle does not have an infinite number of pieces. If things were otherwise we could make no headway because every piece that we discarded as *not* fitting would leave just as many more to examine. Sometimes in exasperation we say: "I've tried *every* piece in that corner," but we still feel sure that we have not. We assume that the number of pieces in the puzzle is *limited*.

The analogy with the picture puzzle is dangerous, like all analogies, and must not be pressed too far. But one other feature is extremely interesting. Do you remember that when you first start on a puzzle you may have only a very vague idea of the completed picture? You say that this looks like a piece of sky, and that other like part of a coat. Your guesses are wild. You group together all of the light blue pieces and the more of them you are able to fit together the clearer your conception of the picture becomes. Yes, this must be sky. See the branches of the tree. Ah, a woodland scene! You have developed an hypothesis. But later your companion sings out that the part he is working on has a fireplace in it. It cannot be a woodland scene. Toward the end you discover that the sky is seen through an open door. The picture is an interior! With each piece fitted into the whole your hypothesis about the nature of the picture becomes more secure, and it probably undergoes several changes before you are finished.

And the same is true of our study of nature. Man's first guesses were very crude. Thales said that everything is made of water. Water could be solid, liquid or gas; living things

needed water; and so on. Later Greeks had four elements; earth, air, fire, and water. Still later came the atomists. Our hypotheses about nature, like our guesses about the picture puzzle, become more and more refined as we proceed. Aristotle would need only to go through a modern grammar school to learn things about nature which would amaze him. We improve our laboratory instruments and we improve our thinking; we gather more and more data and we organize it more clearly. Little wonder that our hypotheses are far more probable than those of the early scientists who started the work on the puzzle of nature.

How Are Inductions Possible from the Single Experiment?

A few years ago a scientific expedition traveled several thousand miles just to be on hand at a total eclipse of the sun and take a few pictures. They wished to test the curved-space hypothesis of the Einstein theory. How could a picture or two supply enough data to test so basic a theory? To-day a single photographic record of radio-active elements tells us more about physics and chemistry than a thousand experiments did in the middle ages. How does this come about? Do you want to know whether or not this patent medicine is injurious to health? Send it to a laboratory and in a few hours you will have the answer. The laboratory technician does not have to make exhaustive tests of the actual effect of the medicine on dogs or cats or humans, he makes a simple chemical analysis and gives an immediate answer. How is this possible? Why, in the later stages of scientific research, is the single experiment so much more significant than it is during the early stages?

Remember the picture puzzle analogy. When you are work-

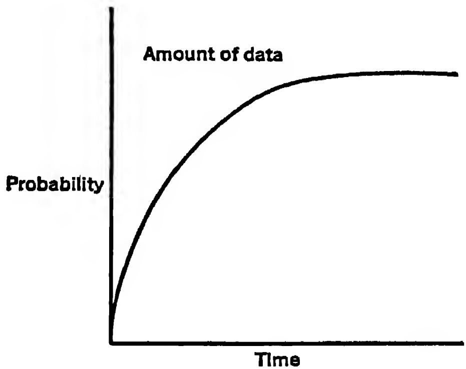
ing on such a puzzle your first attempts to put parts together have to be repeated many times before the right piece is found. You have many pieces from which to choose, and you are unfamiliar with their shapes. Your later attempts to find the right piece for a certain position are much more successful. You find it quickly, for now you have only a few pieces left to fit in, and you are more familiar with them. You see a curious opening and say: "Oh, I saw the piece for that place just a moment ago!"

When you and I were walking through the English countryside studying swans, we assumed for the sake of the argument that we knew little about birds. Knowing little, we had to make a fairly exhaustive study before we could announce with confidence that swans are white. And even then we were wrong. But an ornithologist, encountering a new species of bird, would need to take only one look at an individual before asserting that there probably are others of the same species in the vicinity and adding that in other parts of the world this species might be differently colored. He has had a lot of experience with birds. He knows that a color pattern in feathers is repeated many times. He also knows that peacocks are usually green and blue, but that there are white peacocks. In other words, he is able to fit his new information into a system of knowledge about birds. And, similarly, the chemist who analyzes your patent medicine understands already the physical effects of various chemicals and he needs only to know its constituents to predict what will happen when you take the stuff.

The Scientist Is Erecting a System

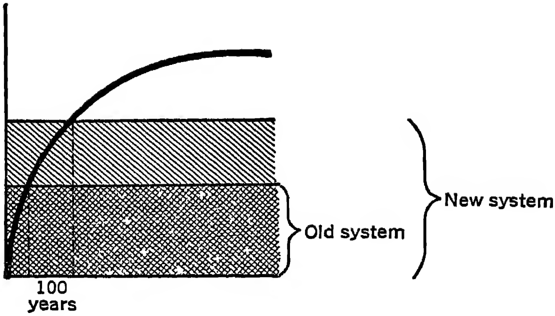
When working on a puzzle we try to fit our pieces into a meaningful picture. The scientist is trying to fit his data into

a coherent and logical system. At first we have only a vague sense of the picture we are constructing. And at first the scientist, having little data at his disposal, will invent a crude system for nature. But the more his data the better his system. Every scientist stands on the shoulders of those who have preceded him. To-day a scientist does not, like Archimedes, have to take his problems into the bath tub, or, like Benjamin Franklin, fly kites up into thunder clouds. The early scientists performed innumerable experiments, and worked out the broad outlines of nature: to-day's scientists may find one experiment momentous. If we represent the advance in our acquisition of data by a curve, we can make an interesting graph of the relation of the data to the system developed. The curve will look something like this:

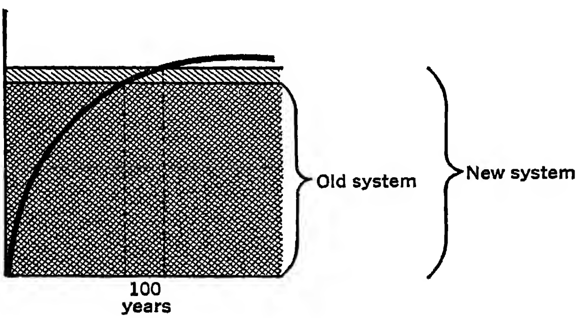


and the area under the curve may be made to represent the system of scientific knowledge developed on the strength of the data.

During early periods a century would bring a large amount of new data and systems would change rapidly:



Harvey's discovery of the circulation of the blood was a major advance in physiology. So was Pasteur's discovery of bacteria. The theory of atoms was a great step forward in physics. The Copernican theory completely altered the thinking of the astronomer. During periods of such radical advance men were not so secure in their hypotheses, though, knowing less about the logic of science, they were willing to burn men at the stake for disagreeing with them! To-day men have increased the probabilities of their hypotheses greatly, but they are more humble. To-day systems change less rapidly and less radically:

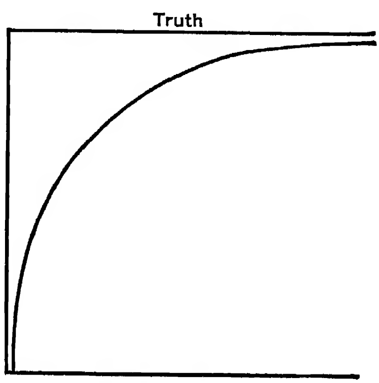


Einstein has superseded Newton, but the change which he has made in our figures is negligible until you start working in light years. Einstein's measurement of the distance from New

York to Boston is more accurate than that of Newton, but so little different that we continue to use Newton's straight line because it is easier to handle. The discovery of radio-active substances was very significant, but they fitted nicely into the Table of Atomic Weights, and did not produce so great a change as the discovery of the atom had. The work now being done with filterable viruses may be of great service, but the service done by Pasteur was greater.

Induction and Deduction in Science

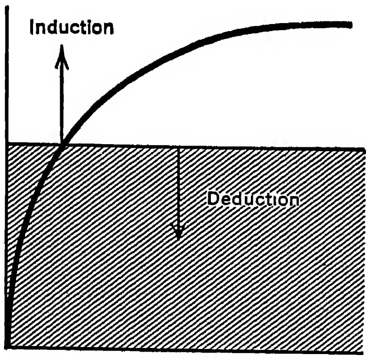
The greatest difference between working out a picture puzzle and studying nature scientifically is that eventually you complete the picture, while man's study of nature will never be complete. The shape of the curve we have drawn is highly significant. Suppose we imagine a finished and certain knowledge of nature in the mind of a divinity, all there is to know about nature's laws. At this point probability would have ceased and certainty taken its place. This divinity would not be bothered about *hypotheses* concerning nature: he would possess *truths*. If we indicate his knowledge on our graph we get interesting results:



The curve of our acquisition of data approaches a level at which all information will be at our disposal. But we have only to imagine ourselves a thousand times smaller or a thousand times larger than we are to realize that the limitations of our knowledge of nature are the limitations of our microscopes and telescopes, and that we shall never have The Truth about nature. *The curve will never reach the top of the graph.*

What might we not find if we could reach out and touch the moon, or if we could get inside an amoeba as easily as we now walk in the front door of a house! Nature will always be something of a mystery to us. We may rest assured that students a hundred years from now will find antiquated many things taught in our science classes to-day. The nature of our curve shows that what we know now will not be as out of date as what people knew a hundred years ago is to us. Surely we shall not be thought as ignorant in 4000 A.D. as the Greeks seem to us. But nevertheless we *are* ignorant, and our "knowledge" will be surpassed. Of that we can rest assured.

But our judgments about nature are more probable and our knowledge is more systematic as we proceed. The logician's way of saying this is to say that science is becoming more and more deductive. Within the system which we have developed we can be deductive: as we advance we employ inductive methods:



Knowing what we do about birds we can deduce when we see a robin that if we find its nest we shall find a nest of a certain construction, containing light blue eggs at one season of the year. And we know what will happen to those eggs. *Within the system we have developed*, we may regard these as "truths" about robins. But when it comes to homing pigeons and their extraordinary flights we develop many interesting hypotheses without being able to fit them securely into the system of our knowledge. We are still dealing inductively with pigeons. In short, without finally becoming deductive, our knowledge of nature takes on more and more the character of a deductive system. With more and more highly developed a system at our disposal we are able to experiment more and more intelligently. Everything we now do in the laboratory is seen in the light of a thousand connections already "established" by previous experiments. Imagine a bushman from Australia brought into a modern research laboratory, and you will get the meaning of this.

Two Reasons Why the Knowledge of the Scientist Will Always Be Hypothetical

Within the system we proceed deductively, but we never know when new data will bring about an alteration in the system and force us to change. *Seen as a whole*, our knowledge of nature is still hypothetical. And it is hypothetical for two reasons.

The first of these is that we can never be certain that our analysis of the available data is correct. Some factor which has been completely disregarded may prove to be of primary significance. Until a few years ago doctors completely disregarded the factor of bacteria. They had not even dreamed of the possibility of these tiny organisms. How many are the

factors in nature of which *we* do not even dream? Or perhaps the factor which we have isolated as of major importance in a given problem has not been sufficiently analyzed. It may be much more complex than we imagine. The primitive tribes that beat tomtoms to drive away the dragon who was eating chunks out of the sun during an eclipse were faced with a complexity of astronomical motions that would have staggered them. What will *we* find when we have further analyzed many of the factors which we now accept as "causes"? No man knows.

The second reason why our knowledge of nature must remain hypothetical is that at its foundation it is based on sheer assumptions. We shall never know whether or not nature is uniform: we can know only that it is convenient to suppose that it is. We shall never know whether or not in reality there are such things as causes: we only know that to suppose causes has given us an extraordinary intellectual tool by which to manipulate the environment. And we shall never know that nature exhibits but a limited variety: we only know that our experiments would be meaningless if we did not think this to be the case. We can never know that these assumptions are true: we can know only that they are helpful.

The function of natural science is to enable us to control our environment. In carrying out this function all three of the above assumptions have been significant, but the one that deals with cause and effect has been most far-reaching. It assumes that the causes of events are in nature, not above nature. It assumes that you can understand the present completely by studying the past. The present comes out of the past and the future comes out of the present in a determined and impersonal manner. Nature does not have choices: if it did there would be an element of freedom which would invalidate the inductive procedure. Nature is not arbitrary; it obeys objec-

tive laws. As Huxley put it, if you could know *everything* about a man and about his environment you could prophecy with certainty where he would be and what he would be doing ten years later. This is the broad picture we get of nature from science, and from the point of view of manipulation it is the most useful that man has ever developed.

But many other assumptions about nature have been proposed. Some suggest that everything in nature is controlled by God. This idea is often useful. Kepler held that a benevolent God would have made the laws of nature simple: he sought for simple laws governing the motions of the planets—*and found them!* Others suggest that nature works purposefully and organically, rather than blindly and deterministically. Aristotle said that the great force in nature was an Entelechy, an urge which looked to the future rather than coming out of the past. This vitalism is strong to-day. Alexis Carrell finds human biology so mystifying that he is willing to endow each unit cell with a personality. Some go so far as to say that electrons have feelings and volitions; that they jump orbits because they get bored. Some of these assumptions are scientific, some are theistic, some are agnostic. Which are true we shall never know.

The "Nature" of the Scientist Is a Tool. How Shall We Use It?

Nature is a concept, a name for a mental system developed by man for a specific purpose. Man wants to live longer and more abundantly, he wants to avoid hunger and cold and sickness, and he has found that he can make progress toward these ends by conducting experiments and slowly thinking out a conceptual system of ideas which shall act as a sort of tool. "Nature," and all the word stands for, is as much a tool as

a hammer or a saw. Like all mental tools it is one which is constantly improving. There is the "nature" of Thales, of Democritus, of Aristotle, of Galileo, of Newton, of Einstein. No one knows or can imagine what the "nature" of the future will be. The tool we now possess may in time seem extremely crude, as crude as the hand shovel is when compared with the steam shovel, or as a camp fire compared with an oil furnace.

But man has other purposes and other systems. He not only wants to live more abundantly and longer, he wants to lead what he calls the good life. He seeks aesthetic values. He seeks wisdom. Science is logically hypothetical. It can say that *if* you want to achieve a certain practical end, experiments will tell you with high probability what you had best do. But what are the ends of life? We wish to avoid hunger and cold and sickness, but is that all? Do we live for physical well-being? To this question science can give no answer. It does not deal in values. Some believe that "nature" describes all of reality, that there are no values. Some believe that "nature" describes that which is entirely unreal and unimportant. Some believe that "nature" describes a part of reality, to be understood as part of a whole. This is the major problem of philosophy, "What is reality?", a problem which each of us is attempting to answer every day. But philosophic problems do not have final answers.

We have described the logic of our study of nature and have tried to emphasize its practical importance in the study and manipulation of our environment. There are none, no matter what their philosophies, who will deny that it is important to study the environment, that it is important to develop scientific knowledge. Hence the significance of a study of the logic of the scientific method, of how to seek causes and uniformities, and of a realization that the judgments of science are always hypothetical. But we shall not fully understand

natural science in its structural aspect until we know more about the structure of conceptual systems as such. Though science never reaches this goal, it is always approaching a completed deductive system. We shall now turn to the study of deductive systems, and much that we shall say will apply to the field of scientific experimentation which we have just been investigating.

PART THREE

The Structure of Deductive Systems

Chapter Eight

HOW GAMES ARE MADE

A FEW years ago a friend of mine was traveling in Soviet Russia. He is a lover of games and rather proud of his ability at the checkerboard. One day he was visiting a Recreation Park near Moscow where a number of Muscovite gray-beards were playing games of checkers. No sight could have pleased him more. He rushed over to the tables and, knowing no Russian, issued a challenge in sign language to one of the old gentlemen. The challenge was accepted with a gracious smile and bow, and soon the international match was well under way. My friend played his usual masterful game and was soon aware that he was in complete control of the situation. He had manœuvered himself into a position from which he would soon sweep the board in triumph. This was really too easy to be much fun. But curiously enough his opponent seemed equally content. It almost looked as if *he* thought he was winning! Not a suggestion of worry showed on his face. Perhaps he was a beginner. Then suddenly, to the consternation of the American Expeditionary Force, things began to happen. The Russian picked up one of his men and calmly zig-zagged back and forth, forward and backward, and in one move captured nearly all of my friend's pieces! "Look here," he shouted, "You can't do that! Only a king can move backward." Not comprehend-

ing, of course, the "victor" offered a handshake full of sympathy and folded up the board.

It took an interpreter and many well-chosen words to handle the ensuing international crisis. In the Russian game of checkers, it seems, the men *can* move forward or backward. This was not a subtle piece of communist propaganda, an expression of dissatisfaction with the *status quo*. The game of checkers is like that. We somehow get the idea that a game we have played all our lives was handed to men on a silver platter by the Divinity of Recreation, and the rules inscribed forever on tablets of stone. No, games develop. And they develop differently in different countries. There is a Polish checkers game that is played on a board of one hundred squares. A piece when crowned king can move forward or backward on rows, files or diagonals. The Canadian game is still another. It has one hundred and forty-four squares and employs thirty men to a side. In the Turkish variety a piece may move forward diagonally or sideways, thus employing both colors of squares on the board. Better find out the rules before you get embroiled in an international match!

The development of a game is a matter of long experience. Men must first have played very simple games, say with stones placed on a rough "board" traced on the ground. Perhaps two men were amusing themselves one day, making up rules as they went along. When they had worked out something they enjoyed they probably excited the curiosity of others, and the game was soon played all over the village. Then some bright fellow discovered that the game would be more interesting with a larger board, or fewer stones, or stones of different sizes. When the movement reached the dimensions of what might be called a "fad," artisans began to make boards and carve pieces, and the great Game Industry had begun. Other games developed. Some were played with round objects hit with sticks,

and turned into baseball and cricket and tennis, and so on. Some were played with little pieces of paper, and after a time we had everything from bridge and poker to solitaire and slap-jack.

All games have changed and most are still changing. The game of baseball played in Big League parks to-day is a very different game from the one in which the catcher stood back and caught the pitch on the bound. Within a few years we have seen the change from auction bridge to contract, and now someone with diabolical or financial intent has introduced a fifth suit and a new version of the Great American Pastime.

The Elements of a Game: I. A Field and Counters

All games are patterns within which men amuse themselves. Their structures have evolved with steps as definite as the steps in the evolution of Man. And, like men, who all have heads and arms and legs and so on, all games have the same general structural elements.

First of all, you must have some kind of a board or field to play on, laid out in a design, geometrical or otherwise. The checkerboard design, using two colors, is one of the simplest and is employed for a great many games. The Chinese game of Go uses a board nineteen squares on a side, but the squares are all the same color and the counters are played on the intersections of the lines. Another design often employed is that found in parchesi, a series of steps leading from a Start to a Home. The more recent Monopoly uses this design. The design of bridge is the familiar North-East-South-West. The design of poker is a circle of variable diameter. The fields for sport games also fall into various classifications. Soccer, lacrosse, hockey and basketball are played on one kind of field, which is

divided up into a "gridiron" for football. The square design of baseball, built around four "bases," is fairly unique. The design in cricket, involving two "wickets," is simpler. Whatever the game, it will be played on some board or field. Sometimes this element of structure appears only in the course of the game, as in the popular solitaire, Canfield. In others, as in rolling dice, the board or field is marked on the men. Instead of throwing men on a board of six squares, you "throw" the six squares on the men.

Once you have decided on the board or field of play, the question arises as to what kind of men or counters you are going to use. In sports the answer is simple. You usually play with a spherical object, large or small, heavy or light. And generally you hit it with bat or racquet, though sometimes the toe or the head or the hand is employed. Sometimes you catch it in a basket or a net or a glove, and then throw it again. In checkers we play with twelve men which are exactly alike in appearance and power until they reach the eighth row, when they become kings. Much more complex is the set-up of men in chess. In that game you have six different kinds of men, each with a different move. In tit-tat-toe you play with crosses and your opponent plays with circles. Perhaps the most familiar set of accessories is a deck of cards. With these divided into four suits and with thirteen different cards in each suit, arranged in order, the possibilities are many. Sometimes you play with cards and pegs, as in cribbage; sometimes you play with cards and chips, as in poker. But the difficulty of a game seldom depends on the properties. A pack of cards is fairly complex, yet the game played with it may be very simple. The counters used in Go, lozenges of two different colors, are very simple but the game probably the most complex ever worked out by the ingenuity of man.

The Elements of a Game: II. Rules

Now we have a board and counters. The next question we ask is, "What shall we do with the counters and the board?" Shall we start with them off the board, as in parchesi, or with them on the board, as in checkers? When we get them on the board, how shall we have them move? In opposite directions, as in checkers or football? Or around in a circle, as in bridge or Monopoly? Shall we allow the element of chance to come in, by throwing dice or spinning a wheel, or shall we place entire reliance on skill? What will be the object of the game? To get all of the counters "home"? To score the largest number of points? To capture the opposing players? To command the greatest amount of area?

The rules in chess are fixed by international agreement and are extremely rigid; the rules of football are decided upon and altered annually by a rules committee, and are so complex that referees and coaches often go to a special school to learn them. One has only to take a wife or sweetheart to a football or baseball game to realize the importance of rules. Sometimes the rules are informal, as in the game of anagrams, where they vary from family to family. Often you can get a highly amusing set-up by reversing the rules of some familiar game. Hearts is the reverse of bridge, for you try to take as few tricks as possible. There is a game of cut-throat chess in which, instead of trying to capture your opponent's king, you try to force your opponent to capture your king. But this is not always possible: the reverse of football could only be a track meet.

The Elements of a Game: III. The Combinations

When you have your board, have placed the counters on it, and know the rules of the game, you are ready to play. Now the various possibilities of the game appear and the fun begins. The game you play most will depend almost entirely on the kinds of combinations you enjoy. These combinations, or developments, *are* the game.

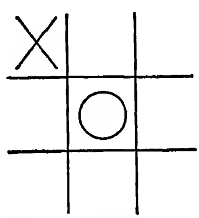
In bridge the combinations are tricks; in poker they are straights, flushes, full houses, etc. In football they are plays and defenses. It is said that there are five fundamental combinations in checkers on which the strategy of the game is based. In chess there are many more, ranging from opening gambits to various types of end game. A champion will spend a lifetime memorizing them and writing them down in books. The combinations in Monopoly are based on transactions in real estate. The combinations in anagrams are found in a standard dictionary. In most games of solitaire the combinations are based on runs of alternating red and black cards, as in Canfield. In dice games the combinations are simply sums of pairs of numbers from one to six, arranged in certain orders punctuated by sevens and elevens. In tit-tat-toe the combination desired is three-in-a-row: in the somewhat more complex Go-Bang it is five-in-a-row. Parchesi gains zest with its blockades and safeties. Cribbage adds the well-known fifteen-two, fifteen-four, fifteen-six, and so on, to the poker combinations.

One of the interesting things about combinations is that you can take the same general kind of men, play on the same field, employ almost identically the same rules, yet work in terms of different combinations. In the game of Rummy you use an ordinary bridge deck, are dealt seven cards, and try to draw and discard so that you can lay your hand down on the table in

trios of the same value or runs in the same suit. The recent innovation of Lexicon cards has shown that you can play a Word-Rummy that is almost identical with the older game except that the cards have letters on them instead of numbers, and the combinations which you lay on the table are words of two or more letters. You can take these cards with letters on them and play a Word-Canfield. The cards are laid out exactly as in the regulation game and the procedure of playing through the pack is the same, the only difference being that the cards are not discarded in suit sequences on aces but rather in terms of words formed. It would be perfectly easy to make up a game of word-poker. And there might be some interesting possibilities in word-bridge.

Various Characteristics of Games: Skill *vs.* Chance

What kind of game do you like most? Games may be roughly divided into games of skill, games of chance, and games which combine the two. Games of skill may be simple or complex. Tit-tat-toe is a game of skill, but so simple that all of the possible combinations can be worked out in a few minutes. If, for example, you start with a cross in one corner, your opponent *must* put a circle in the middle if he is to prevent you from winning:



Otherwise your next cross will be in the opposite corner, he will then have to play the center in order to prevent three-in-a-

row, and then you can put a cross in one of the other of the remaining corners and gain two winning possibilities which cannot both be canceled in one turn. More complex is Go-Bang, played on an unlimited field of squares, in which the object is to gain five-in-a-row. If you have any interest in skill it is a far better game. If you enjoy skill, but not too much of it, checkers is better than chess. If you are interested in the serious study of a game, chess is better than checkers. But even in chess there are limitations. If you play the game day and night for ten years you will have the most important combinations in mind. Most intricate of all, and depending entirely on skill, is the Chinese Go. A serious Go player has to give his life to it, and the players are ranged in seven ranks of ability. A decent game cannot be played in one sitting.

There are games of manual skill also, such as baseball or tennis. All sports emphasize manual dexterity. Some, like trap-shooting, are entirely dependent on it. But in most there is a factor of mental skill which shows itself in strategy and tactics. A good baseball player is a student of the structure of the game, knows when to play in close, when to pole out a sacrifice fly, when to bunt. A good quarterback shows exceptional mental skill and adds appreciably to the fascination of football. Whenever mental skill enters the situation there is a study of the structure of the game. A good tennis player who knows how to mix his chops and his drives, how to place his shots, how to manœuver his opponent out of position, will beat a better technician who uses no strategy. Most people have a certain amount of intellectual curiosity, and hence like best a game in which study and strategy are possible.

Some people place so much emphasis on skill that, like the bridge addicts who turn to duplicate-bridge, they will try to rule out every element of chance. Others much prefer games of almost pure chance. It is difficult to rule out skill altogether.

Some solitaires, like Idiot's Delight and Canfield do it fairly successfully. A mathematician or a psychologist will play a skilful poker game, but usually the chips are piled in front of the "lucky" player. "Lucky at cards" is a familiar description. The game of bridge, as played after supper in the dormitory or frat house of the average college, is largely a game of luck. Games of chance are the most popular gambling pastimes; the horse-race game played on shipboard, in which the horses advance according to the throw of dice, and the slot-machine games, are perfect examples. Unless played over long periods of time, more card games are won because of the hands dealt than because of the skill shown. This is not to say that because the factor of skill is subordinate these games of chance do not have a structure. They all have structure, all possess the elements necessary to every game. The difference is that in games of chance structure is not emphasized because attention to it does not help much in winning. The outcome is in the lap of the gods, and all players regardless of ability have an equal or nearly equal chance.

How to Make a Quick Fortune

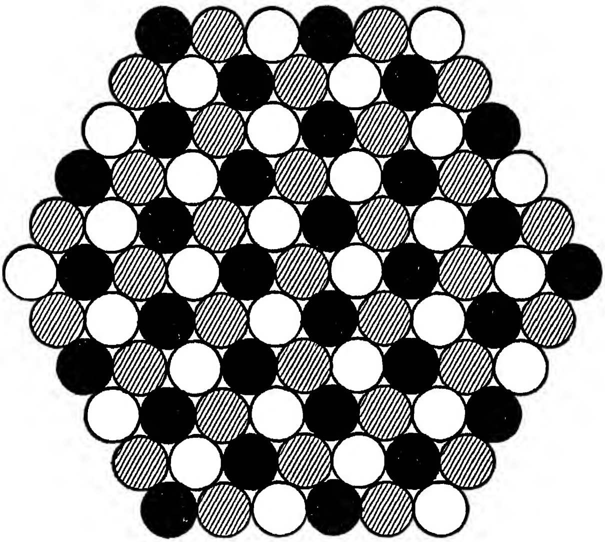
Would you like to make an easy million dollars? If you are really interested I can show you how to do it. Games have to be invented, and it is hardly conceivable that all of the best games that men will ever play are already devised. Why not perfect some new game that will catch the fancy of the public, cost fifty cents to make, sell for a dollar, and be sold to one person in every fifty in the country? All you have to do is to find a new board, new counters, think up interesting rules, and show that the possibilities are fascinating. If you are lazy and would be satisfied with a few thousand dollars you have only to take an old board and base a new game on it. Some man was

bright enough to take the familiar parchesi board and make a novel real estate game out of it. This sort of thing is done every year, usually around Christmas time.

If you will cut me in on the royalties I will give you a start toward your treasure island. In order to sell the game widely you will want to invent one with a wide appeal. You will want to appeal to those who like to make a study of a game, and to those who wish to play for the careless fun of it. You must appeal to those who are proud of their intellectual capacities and want to show off by playing masterfully. Hence skill must enter. You must also appeal to the shallow pate who is bored with a serious game, and who is glad of the opportunity to hide his incompetence behind the skirts of chance. From the point of view of selling the game the last of these appeals is far more important than the others. You will want a game whose rules are not too complex for people to remember. I am willing to guarantee that a game like chess will never sweep the country. Yet you want a game which does not err on the side of being too simple, like tit-tat-toe.

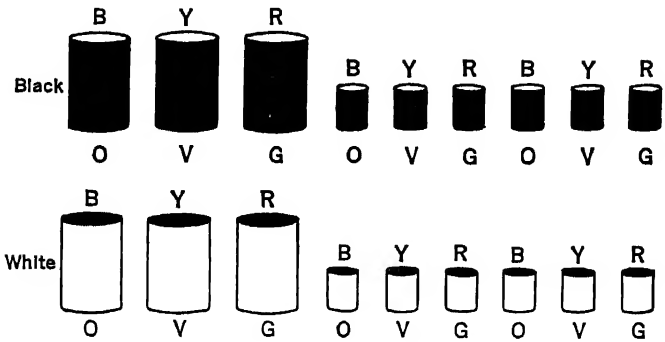
From many points of view Go-Bang is a perfect game. The rules can be learned in less than one minute, yet there are possibilities of complexity which extend as far as the amateur or
guarantee that a game like chess will never sweep the country.
because it is a pure game of skill. People do not like this. They are human and practically all humans are fascinated by the factor of chance. Bridge really is a perfect game. You can play it with $99\frac{44}{100}\%$ of your brain shut off, or you can puzzle hours over intricate problems and situations. Those who like to exhibit their abilities can show considerable skill, yet the excitement of what chance may bring with every hand dealt is fairly intoxicating. These are the principles you must follow, but don't forget my royalty as part of the overhead of your venture!

This prospect of royalties excites me. I'll take you a little farther toward the new and wonderful game. Has it ever occurred to you that no game has yet been worked out to be played on a board of *three* colors instead of the conventional two? Now it just happens that there are three primary colors; red, yellow and blue. How about a hexagonal board (we can call the game *Hex*—patent not yet applied for!) like this?



This is the first step, and in some ways the most difficult one. We have a board to play on. If you have any imagination at all you will see immediately a number of hexagonal and triangular patterns on this board which might be used in various ways. I hope you will get even more excited when you notice the unique advantage of a hexagonal board: two, or four, or six can play at once. The now "old-fashioned" checkerboard generally seats only two, though there is a very good four-handed checker game, played with partners. But it will not accommodate six. Hex marches on!

The next step is to decide what kind of counters shall be employed. For my Hex board I have made a set of men that looks like this:



Notice a number of things about this set of men. They are of two different sizes: the small ones can be given one type of move or power, and the large ones a more rapid or diverse transit and greater power. This difference allows for an element of strategy, yet is not too complex. The colors employed on the ends are the six colors of the color circle (*Hex is Educational!*—advt.), and the two colors on the bases of any one piece are complementary, e.g. red is opposite green. The various ways in which these six colors can be combined with the three primary colors on the board offers many possibilities.

The rules? I have tried about a dozen different sets of rules on my Hex board. Some have given fairly good results, others not so good. But you make up your own rules. After all, you must do something to earn your royalties. When you have done a little experimenting you will find that the problem of balance of forces is a hard one to solve. You must not give the first player too great an advantage. On the other hand, you must not make the offense too weak, or the game will never end. The most ingenious thing about chess, for example, is the balance

of power at the beginning. A seemingly slight rearrangement of the chess men would give either White or Black too great an advantage and spoil the game. And don't forget to introduce the element of chance, with dice or a spinner or some other such device. Good luck to you!

A Game Is a Logical Thing

One of the curious things about most games is that they *can* be played without the usual paraphernalia of boards and counters. Blindfold chess illustrates this clearly. In this case the entire game is played mentally without the help of either board or men. Two college friends of mine used to amuse themselves in class playing chess simply by writing their moves alternately on a slip of paper. A good bridge player can run through a hand without so much as touching a card, in fact he does so in working out a bridge problem in his head.

The reason for this is that all games are mental structures, and most of them exclusively so. Sports, football and baseball and the rest, involve a factor of physical prowess, and hence have to be played physically. You cannot imagine a football game played by twenty-two huskies seated blindfolded in a drawing room. But most games which do not depend on physical skill can be played in the mind. These games are entirely logical in character. The boards and counters usually employed serve only as material symbols which make the game easier to visualize. But these material aids are not essential. The "move" in chess and the "play" in bridge are actually mental events, logical happenings within a logical pattern, and the advancing of a pawn or the trumping of an ace is merely the outward and visible sign of what has taken place. You can illustrate this easily for yourself by indulging in a game of tit-tat-toe without pencil and paper. The things you pay good money for at the

game-counter of a store are to the game what properties are to a play, visual aids. The Chinese play without properties is the dramatic equivalent of blindfold chess. A game is a set of ideas in logical relationship to one another.

This explains why a study of games is a logical study. Logic concerns itself with mental structures, and games are both mental and structural. Those who are fascinated by structures are great lovers of games, and more especially of games of skill, for in the development of skill the structural aspect of the pastime is emphasized. How many trumps are still out? Can I manœuver this checker player into a pair of Dutchman's breeches? Shall I save for a straight or a flush? Shall I build with the three or the five half of the domino? How can I make a new word out of "streamer" by adding one letter? Shall I peg a fifteen-two or a pair? Can I get the knight's pawn into the eighth row in four moves? A game is a battle played on a certain terrain, with a definite army of forces, and according to rules to which both sides agree. It is a process of stepping into an imaginary world with your opponent and trying to outwit and outplay him within the restrictions imposed by the logic of that world. To the lover of games and structures there is no point in cheating. It is great fun to win but even the loser enjoys the intricacies and possibilities of the chosen structure.

The Inventor of a Game Is a Writer of Logical Fiction; The Player of a Game Is an Actor in a Logical Play

When you sit down to work out your million-dollar game you will start at scratch. You can employ any board, any men, and any rules. You are a pioneer. You will have to do considerable experimenting, make many trials and commit many errors, before you have developed a satisfactory structure. In this ex-

perimental procedure you will be proceeding *inductively* to create a deductive system. We have seen that the inductive procedure is one in which by means of experimentation with various hypotheses you approach a deductive system. The difference between the scientist and the inventor of a game is, of course, that the scientist is trying to work out a deductive system which will describe the structure of his environment, objective in relation to him, while the inventor of a game is allowed to work out any kind of a pattern so long as it makes a good game. The scientist is like a detective who is trying to find out what *really* happened at a certain time and place, while the inventor of a game is like a novelist, the teller of an imaginary story.

Part of the charm of a game, as we all know, is that the events are *not* real. In a war game the Blue forces "destroy" the Red, but as soon as it is over the Red ships arise from their watery graves and the Red forces come to life again. In the Middle Ages it was one thing to be beaten by real knights and bishops and another to be checkmated by chessmen. A sound drubbing in bridge never hurt any one. It's all part of the game, as we say.

When you and your friend sit down at the game board, however, the game is already defined as a system and you both subscribe to its limitations. You move into the world of a deductive system. It is as if you had decided to take parts in a play in which you are rivals for the hand of the heroine. You do not know who will be the lucky one. The author has not finished the story. But you are cast as the rich man's son and your friend is cast as the son of the bootblack, and within your two rôles you have to do what you can to bring about the ending you (*in your rôle,* of course) desire. When it is all over you discard the costumes and are friends again. But on the stage, or across the game board, you are bitter enemies. When you play a game with your friend you move into a deductive wonder-

land, and the fun you have depends on how wonderful the new land is. Playing tit-tat-toe is like living in a one-room apartment: playing chess is like moving into a medieval castle.

Structures for Amusement and Structures for Use

An officer in the Japanese army must reach a certain rank as a Go player before he can receive a higher commission. It is often said that familiarity with the strategy and tactics of chess will help in the conduct of a real military campaign. Many game structures are useful as well as recreational. In fact, many games have been developed out of structures originally built for more useful purposes. The game of anagrams, for example, uses the structure of words, and words are primarily instruments of communication. Chess has obviously borrowed from the structural pattern of warfare, with its defenses and offenses, officers and privates, queens and kings. The game of poker owes its pattern largely to the number system, involving a series of numbers running from one to thirteen. Every good poker player knows that the scoring in the game is worked out by careful consideration of the mathematics of probability. Dominoes is based on the same mathematical structure.

It is only a short step from recreational patterns to useful ones. The step is so short that there is hardly a useful pattern which does not provide recreational opportunities. From the structural point of view there are no patterns more interesting than those which come under the general head of Mathematics. Because of their great importance to us, arithmetic and geometry have been worked out with unusual care and in considerable detail. In any attempt to study structure they are most helpful. Hence we shall now turn in their direction.

Chapter Nine

MATHEMATICS AS A GAME

ONE test of an interesting structure in a game is whether or not it lends itself to puzzles and problems. Bridge meets this test well. Many newspapers and magazines publish bridge problems regularly. The continued popularity of the cross-word puzzle shows the fascination of the structure of words, as does the less familiar but even more captivating cryptogram. Chess offers all kinds of puzzles, of which the most famous because the most difficult is that of covering the entire chessboard with a single knight and not repeating any squares. It is called the Knight's Re-Entrant problem. What is the largest number of kings you can put on a chessboard so that no one can take another? What is the smallest number of bishops necessary to command all of the squares on the board? More familiar are the "White to play and win" and the "White to play and win in three moves" set-ups.

Some of our best puzzles and recreations are provided by the structure of numbers. Take any number from one to ten. Multiply it by five, add seven to the product, multiply the sum by four, add two to the product, and divide the sum by ten. Tell me your result and I will tell you the number you chose. Can you take a three-pint and a five-pint jar to a spring and bring back exactly four pints? Then there is the medieval problem of the fifteen Turks and fifteen Christians caught in a storm at sea. In order to save the ship and crew, fifteen of the company

must be thrown into the sea. Accordingly the passengers are placed in a circle, and beginning at a certain point every ninth man is cast overboard. How arrange all thirty passengers so that the Christians can be saved? There are the digit problems. If 3333 digits are used in paging a book, how many pages has it? Can you find the digits in the following division?

```
a b) c c d a b (b c b
     c e b
     -----
         f a
         g c
         -----
         c e b
         c e b
         -----
```

Completely fascinating to those interested in structure are magic squares, in which the sum of the numbers in every row and column and in the two diagonals is the same. Here is a simple magic square employing the numbers from 1 to 16:[1]

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

[1] There is also the so-called "diabolical" magic square, in which even the broken diagonals add up to 34. It has the most interesting pattern of all.

Can you find the pattern in this arrangement? Using the same numbers, can you find another pattern with the same magic properties? The patterns for a square using the numbers up to 25 are more intricate. Can you find one? Benjamin Franklin in a single evening worked out a magic square *sixteen* squares on a side; rows, columns and diagonals adding up to 2056!

The Number System Is Built in the Same Manner That Games Are Built

The structure of the number system is the structure of many games and recreations. No need to tell you, though, that men did not sit down and work out arithmetic for the fun of it. Every child who ever struggled with the multiplication table knows better. Men worked out arithmetic because they needed it, and only later did they discover that one could have fun with it too. Its usefulness cannot be over-emphasized. Try to picture a world without arithmetic. Barter would have to be substituted for money transactions; we could no longer calculate size, weight or speed; we should have to tell time by the position of the sun in the heavens. These are just a few of the implications of the loss. But arithmetic is simply a series of symbols or "players" which we agree shall perform according to certain rules, have certain relationships to one another, and combine in certain ways.

Did you ever stop to think of the difficulty that a Roman schoolboy must have had in working out problems of division? How would you divide MXXIV by VIII? We should be very grateful to those who helped decide that we would use Arabic numerals rather than Roman ones. The division of 1024 by 8 is a far easier problem. The players in our game of arithmetic are the familiar digits from 0 to 9. When we have gone through all ten of them we start over again, putting a "1" in the ten's column

to signify that we have been through them once, and so on. In your automobile the speedometer is so constructed. The units wheel pushes the tens wheel one tenth of its circumference every time it completes a revolution, and the tens wheel does the same thing to the hundreds wheel when it has rotated completely once. If you would like an interesting puzzle you might try to figure out how a Roman would have registered mileage on one of his chariots *using only Roman numerals!*

One of the really absorbing things to ponder as regards the structure of our arithmetic pattern is its accidental character. Consider the derivation of the word, "digit." It happens that we are born with ten fingers. Surely men, like children, first counted on their fingers, and it is probable that the reason why we start over again every time we pass ten units is that primitive men started using the same fingers over again when they counted beyond ten. It would have been quite different, of course, if we had been born with twelve fingers rather than ten. In that case we should almost certainly now be using a duodecimal system rather than a decimal one. We should have twelve digits, and start repeating at thirteen! This would have given us an entirely different pattern. Those who are familiar with numbers tell us that a duodecimal system would be much more convenient than the present system. Twelve is divisible by 1, 2, 3, 4, and 6; ten is divisible by only 1, 2, and 5. But we do not change, probably for the same reason we do not adopt a thirteen month calendar. Be that as it may. We possess a structure which we call arithmetic, and the fact that there is more than one pattern which might be employed for counting and calculating shows that the pattern is a *mental* one, like the patterns of games.

We take these digits, and combinations of digits, and we make a set of the rules according to which we are going to "play" with them:

$$1 + 1 = 2$$
$$1 + 2 = 3$$
$$1 + 3 = 4$$

And so on. $6 + 7 = 13$. How do we know? Because that is the way we have constructed the pattern. If we had been using a duodecimal pattern we should get $6 + 7 = 11$! We soon find that we can reverse these rules of addition, and we have what we call "subtraction." Later we find that we can make a shorthand of addition. Instead of adding eight 3's together tediously we work out a multiplication table and simply say, $3 \times 8 = 24$. Reverse this process and we have division. Later the pattern gets much more complicated. We add decimal points and minus numbers, then radical signs and exponents. Finally we get to stages at which the system ceases to have practical application. For example, $\sqrt{-3}$ is what is called an *imaginary* number. At these points mathematicians are expanding the structure of arithmetic because they are curious about the pattern itself and want to work out all of the possibilities. The thing has truly become a game.

Arithmetics Familiar and Arithmetics Unfamiliar: There Are Many Possibilities

But arithmetic is not just a game. Somehow it describes what goes on in our world. If each car in a garage has four wheels we know that all fifty-seven cars together will have 228 wheels. And we know this without counting, not because some one once decreed that $4 \times 57 = 228$, but rather because there is something in the pattern of the environment that is like the pattern of arithmetic. This can be illustrated in another way. We find that numbers are even or odd, either exactly divisible by two or divisible by two with one left over. This is also true of the

pattern of our world. If we put any number of objects together in one pile we find that that pile can either be divided into two smaller piles which are exactly the same size, or they can be divided into two such piles with one object remaining. Add one to any odd number and it becomes even. Add one object to any pile not halvable and it becomes so.

When I was in school we used to play a game with numbers. This looks queer, but is not half so queer as what we can do if omitting every number that was either divisible by seven or contained a seven. It was fun because it was so easy to make mistakes. But once you caught the pattern of the thing it went fairly well. These numbers, as we recited them, constituted a rather curious number system, not useful, but nevertheless a system. There are a great many such eccentric patterns which might be worked out. Suppose you build a system in which every third number is doubled:

1, 2, 6, 4, 5, 12, 7, 8, 18, 10, 11, 24, 13, 14, 30 . . .

This looks queer, but is not half so queer as what we can do if we really *try* to make up a peculiar pattern of numbers.

We might put it this way. Given a set of "digits" with numerical significances, we could combine them according to any rules whatever and hence develop any number of curious patterns. And with these patterns we could undoubtedly play a whole variety of interesting games. We do not do this, of course. What is the gain, when we already have at our disposal a useful system whose pattern is as rich and as fascinating as man could wish? Who wants to learn an entirely new pattern when he can use just as well the one he learned in school? It is easier and just as much fun to play with the conventional system. But it is worth bearing in mind that the system we learn in school is only one among many which our minds are capable of constructing.

In developing the system of arithmetic, man was faced with a problem very similar to the problem before you in working out Hex. The object is different, to be sure. You are trying to invent a system which will appeal to a great many people: you are trying to produce amusement. He is trying to devise a system which will be useful in the manipulation of the environment: he is trying to produce utility. So you pay considerable attention to what makes a pattern fascinate from the point of view of recreation, and you can use any kind of a board, any men, and any rules so long as you come out in the end with a popular system. He must give considerable attention to his environment. Indeed, it could be said that the environment is the board on which he has agreed to play, and that given that board he can use any counters and any rules, and produce any system that will be useful.

We have seen, for example, that he has tried Roman numerals and Arabic numerals. Experiment has shown that the rules necessary to manipulate a system using Roman numerals are more awkward than the rules necessary to a system of Arabic numerals, so we have favored the latter. And we are now realizing that a set of twelve numerals, let us say,

1, 2, 3, 4, 5, 6, 7, 8, 9, ∩, ʎ, 0

would be more useful than the conventional ten. The primitive system is one in which fingers are used as counters. The rules in such a system are relatively simple. But the system itself is seriously limited. Try to count a thousand dollars using just your fingers and you will get into an awful mess. If you go into some banks in Quebec you will find the cashier equipped with an abacus, colored beads on wires in a frame. This is an improvement over the use of fingers, but its limitations are obvious, too. Try to figure the income on $475 at 3½% with beads!

Arithmetic Is a Deductive System Devised by Man

Columbus *discovered* America. Edison *invented* the electric light bulb. And man *invented* arithmetic, just as you are endeavoring to invent the game of Hex. The difference between inventing and discovering is great. Some logicians have tried to picture us as discovering the truths of arithmetic. How do we know that $2 + 3 = 5$? We take two apples and pile them together with three apples and count the result, discovering it to be five. Then we take two men and stand them up beside three other men and count the row. Lo and behold, five again! And so we go around our world putting two's of things with three's of things and noting that every time we do it the result is five. Hence by a process of induction we develop the hypothesis that two and three make five. The more we perform the experiment the more nearly certain the hypothesis becomes.

Unfortunately there are at least two serious objections to this "discovery" idea. One is that all of the men who have lived on this planet working twenty-four hours a day during the entire span of their lives could never "discover" all of the truths we possess about numbers. For example, we *know* that if we take 20,591 beans and pour them into a sack which already contains 4,378 beans we will have 24,969 beans in the sack. It would be quite safe to wager that no man has ever experimented with that particular sum, no man has ever counted that number of beans or minnows or books or anything else. We know that $20{,}591 + 4{,}378 = 24{,}969$ *because we have devised the system and know how it works.*

A second objection to the "discovery" theory is found in the fact that the truths of arithmetic are indubitable. We saw that when we worked inductively, developing hypothesis, we had to stay in the realm of the probable, both because we based our

thinking on assumptions and because we could never come into possession of all of the facts. We can never come into possession of all of the facts which arithmetic describes. We have not put together two's and three's of all possible objects in our environment, hence if arithmetic were a discovery we *might* some day put two dollars and three dollars together and discover that we had six altogether! That *would* be a discovery! But you might just as well give up hope and stop experimenting with your cash. Two dollars and three dollars will always make five dollars, because of the nature of "two" and "three" and "five." We ought to know. We made them up, invented them. It is just the nature of the system.

The difficulty which my checker-playing friend had with his Russian opponent was due to the fact that they were playing on the same board and with the same men but in different systems. They were acting upon diverse assumptions. Assumptions make a world of difference. When men sit down to play a game they must act on the same assumptions, play according to the same rules. In chess you could never work out a successful strategy to capture your opponent's king unless both of you agreed that kings and their defending pieces should move according to a definite set of rules. The checkmate is a certainty which the loser must acknowledge. If some one comes to you and announces that $6 + 7 = 11$ *he may be right within his own system.* He may be far in advance of his fellows and using the duodecimal system. But his system will not do him much good unless he is a hermit. He could not transact business with any one. And that, of course, is the reason why in school the teacher will without any misgivings or doubt mark $6 + 7 = 11$ WRONG with a sweep of her red pencil. Her job is to teach her pupils the system of arithmetic which men have agreed upon and are using, *the* system of arithmetic at the moment.

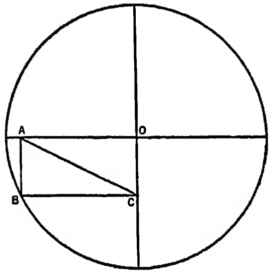
Arithmetic is a system which tells us what we can do and

what we cannot do when we are using it. We know what is right and what is wrong, what is possible and what is not possible, by the simple expedient of examining the laws of the system. No room is left for doubt if the system is well defined. A syllogism is a small system possessing the same characteristic. It plays with classes of objects or ideas, and gives definite rules regarding what can be done with these classes. If one class of objects is included in a second, and the second class is included in a third, then the rule governing such a situation in the system of the syllogism tells us that the first class will be included in the third. Accept the syllogism as a valid system and there can be no doubt that if all of the animals in this cage are cats and all cats have whiskers, then all of the animals in this cage have whiskers. If the system is accepted and the premises true, then there can be no doubt that the conclusion is also true. As we shall see shortly, the syllogism is part of a much larger and more fundamental system, but at the moment we need only to remember that the syllogism gives reliable judgments because it is deductive, and that arithmetic gives reliable truths for the same reason.

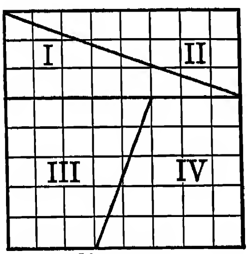
Geometrical Recreations

We have said that a good pattern or system will lend itself to a variety of fascinating puzzles and problems. Many of our best brain-teasers are geometric. Someone may have told you, or you may have noticed, that mapmakers never have to use more than four colors. It is impossible so to relate areas that five colors will be needed to prevent contiguous areas from being of the same color. Try it yourself. And, by the way, if you can prove rigorously that this is so, you will achieve immortality in the field of mathematics. A problem which is perfectly simple when you find the key, but which many good mathematicians

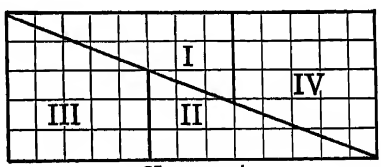
have puzzled over for hours, arises out of the following diagram:



ABCO is a rectangle. If the line AB is three inches and the diameter of the circle is eight inches, how long is the line AC? Most mysterious of all is the puzzle in which you cut the area below on the left along the heavy line, reassemble the pieces, and get an area with one more square in it. You started with an area of sixty-four squares and end with one of sixty-five! [2]



64 squares



65 squares!

Just one more illustration of what can be done with geometry by way of amusement: in a room thirty feet long, twelve feet wide, and twelve feet high there are a spider and a fly. The

[2] Taken from Ball's *Mathematical Recreations and Essays*, Macmillan, 1911, p. 53.

fly is resting on the middle of one end wall, one foot up from the floor: the spider sits in the middle of the opposite wall, one foot down from the ceiling. The spider is not allowed to spin a web. What is the shortest route which he can take to reach the fly? Par for this problem is forty-two feet, but there is a shorter path. Can you find it?

The structure of geometry is one of the most fascinating known to men. It is also one of the most clearly defined of all of the structures he has devised. And, I dare say, our schools being what they are, it is at the same time one of the most familiar. There is no better example of a deductive system.

First Impressions of High School Geometry

Your first experience with plane geometry was probably much like mine. I thought the whole business confoundedly silly. We started with definitions of a number of ideas: "straight line," "angle," "parallel," and so on. We all knew what these things were, but for some unaccountable (to us) reason they had to be defined. Then the teacher, with a great show of wisdom, pointed out certain statements which he called "postulates." Any one could see immediately that they were so obvious that it was unimportant to bother even to state them. "A straight line can be drawn from any one point to any other point": "A straight line can be produced to any distance, or can be terminated at any point": "A straight line is the shortest distance between two points." He then proceeded to enunciate certain "axioms" which were even more ridiculous because more obvious. "Things which are equal to the same thing are equal to each other": "The whole is greater than any of its parts": "If equals are added to equals the sums are equal."

The finishing touch in this comedy of obvious statements came when, with a great attempt to impress us with the mo-

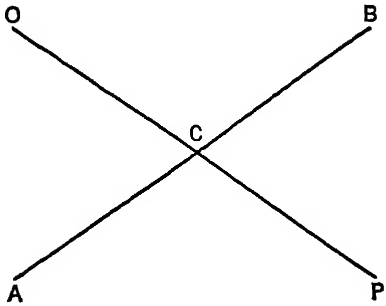
mentousness of what he was doing, the teacher undertook to prove that all straight angles are equal. This was Theorem I in the book. Any one with half an eye or half a brain could see that two straight angles must be equal. Furthermore, why complicate matters by calling these things "straight angles" when they were not angles at all, but lines? As I look back I realize that our poor teacher had a rough time of it that day. When he put the famous Q.E.D. at the end of this first demonstration he acted as if he had done something really important. But we could not see it.

It was not until later in the course that we understood what he was driving at. Some of the later theorems, we had to admit, were not at all obvious. "The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles." "Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum." "The areas of two triangles which have an angle of the one equal to four right angles." "Of isoperimetric regular polygons, products of the sides including the equal angles." Now we began to get something of a thrill out of the concluding Q.E.D. What had happened? We suddenly found ourselves proving things about lines and angles and areas which were not self-evident. And we were *proving* them, proving them beyond the shadow of a doubt. Having noted and accepted the apparently silly and obvious early in the course, we found that we could build on it a fascinating structure of relationships in space. We were getting somewhere.

A Typical Geometrical Proof

Let us see, by way of example, how this geometrical system works. Theorem IV in my book states that "if one straight line intersects another straight line the vertical angles are equal."

This is a good example to use because the theorem is not obvious yet the proof is not complex:



Let line OP cut AB at C.

To prove: $\angle OCB = \angle ACP$

Proof: $\angle OCA + \angle OCB = 2$ right angles. (1)

and $\angle OCA + \angle ACP = 2$ right angles. (2)

How do we know this? The book refers us back to an earlier theorem (II) which states: "If two adjacent angles have their exterior sides in a straight line, these angles are supplements of each other." Then, looking up the definition of "supplements" we find that angles are *supplementary* when their sum is equal to a straight angle. And in the definition of a "straight angle" we find that it is equal to two right angles. Now the next step:

$$\therefore \angle OCA + \angle OCB = \angle OCA + \angle ACP \qquad (3)$$

At this point the book refers us back to Axiom 1, which says: "Things which are equal to the same thing are equal to each other." Good. We have seen that $\angle OCA + \angle OCB$ and $\angle OCA + \angle ACP$ are equal to the same thing, to wit, *two right*

angles. Therefore they are equal to each other. Then comes the final step:

$$\therefore \angle OCB = \angle ACP \qquad (4)$$

What is the reason behind this? Now the book refers us to Axiom 3: "If equals are taken from equals the remainders are equal." Surely $\angle$OCA and $\angle$OCA, being identical, are equal, and when they are subtracted from (3) the results will be equal. And we have proved what we set out to prove.

The interesting thing to note at this point is the various types of authority which we have invoked in proving the theorem about vertical angles. Suppose we list them:

1. Definition of "vertical angle."
2. Definition of "supplementary angle."
3. Definition of "straight angle."
4. Axiom 1
5. Axiom 3.
6. Theorem II.

There would be no point in questioning the definitions: they simply allow us to agree among ourselves upon what we shall mean by the words we use. Definitions help language to be precise and accurate. And, remembering our first reactions to the various axioms stated at the beginning of the book, we should be ashamed to question Axioms 1 and 3. Any fool would accept them! But what about Theorem II? How do we know that it is true? Turning back to it in my book, I find that it has a Q.E.D. at the end, and that its proof depends upon the truth of two definitions plus Axiom 9. But we have decided to accept definitions and axioms. Hence we shall not be disposed to question the proof of Theorem II. Our investigation of Theorem IV has been eminently satisfactory.

Notice again a similarity to the type of argument found in a

syllogism. In the syllogism the acceptance of two premises, for example "All of the animals in this cage are cats" and "All cats have whiskers," allows us to *prove* that "All of the animals in this cage have whiskers." We had found a structure among propositions which allowed us to move with certainty from premises to a conclusion. Once the premises were accepted no one in his right mind would deny the conclusion. The argument proving Theorem IV is more complex, for it involves more than two premises. The premises here are three definitions, two axioms, and a theorem. And the theorem is in turn proved in terms of further premises which are axioms and definitions. But again we have found a structure which allows us to move with certainty from premises to a conclusion. In this second case the structure is not the structure of propositions in themselves, it is the structure of spatial elements. Nevertheless the two proofs are essentially of the same type: premises leading to a conclusion.

The Game of Geometry: I. The Counters

Geometry is like a game. It is a game played with spatial relationships. The counters with which it is played are points and lines and angles (of various kinds) and circles, and so on. The rules according to which the game is played are the axioms and postulates at the beginning of any geometry book, and the various combinations which are worked out are theorems ranging from the simple to the complex. It is one of the most interesting games to study because its structure is so clear.

Consider, for a moment, the counters. Before we get very far in geometry we are playing with right angles, straight angles, acute angles, and obtuse angles. What is an acute angle? It is defined as "an angle less than a right angle." And an obtuse angle is defined as one greater than a right angle. Hence

in this game the meaning of "acute" and "obtuse" may be found by reference to "right." But what is a "right angle"? In my book it is defined as "half a straight angle." So, the meaning of "right" is dependent on the meaning of "straight," and since the meanings of "acute" and "obtuse" are dependent on "right," they are more basically dependent on "straight." But now what is a "straight *angle.*" Further reference shows that an angle is defined as "the opening between two straight lines which meet" and a "straight angle" is defined as one in which the sides are in the same straight line. To summarize.

obtuse (or acute) angle	*defined in terms of*	right angle.
right angle	" " " "	straight angle.
straight angle	" " " "	straight line and angle.
angle	" " " "	straight line.

But in a sense this is only a beginning. What is a straight line? In my book it is defined as "a line which has the same direction throughout its whole extent." It is assumed that we know what "same direction throughout its whole extent" means. But what is a line? It is "the edge along which two planes meet":

straight line	*defined in terms of*	line
line	" " " "	plane

And what is a plane? This is left undefined. We have come to the end of our inquiry.

This is not just good fortune. There *has* to be a beginning somewhere. If you look up "square-rigger" in a dictionary you will find it defined as a certain kind of "ship." And a "ship" is defined as "a vessel." And a "vessel" is defined as "a structure." And a "structure" is defined as "something." And "something" is defined as "a thing." When you are telling your friend about

square-riggers you will say they are ships with certain kinds of sails. And if, heaven help him, he does not know what a "ship" is, you will point and say "That is a ship, and that, and that." And similarly in my geometry book, the very first paragraph has a picture of a block of wood and the text "points" to the six flat faces and says, "These are planes." In the beginning there was The Plane.

One of the interesting things that geometricians have discovered is that you are not compelled to begin with any one counter in this game. For instance, you could just as well say, "In the beginning there was The Point." Then you would define a line as a locus of points, and a plane would be defined as a locus of lines. Even more radical experiments have been performed successfully. Suppose we do not start with points or lines or planes—but with *circles*. Then one circle would amply define a *plane*, the intersections of two circles would enable us to determine the *line*, and the intersections of three circles would give us the *point*. Some geometricians do not bother to have only one undefined term: they use point *and* line *and* plane as undefined elements. The only condition which they all accept, because they must necessarily do so, is that some terms or counters shall be undefined.

If some one tells you that you look like Henry Handsome, you will not know whether to be pleased or insulted if you have never seen Henry. And if he lives in China there is not much chance of finding out. But a friend just back from China may know him and say he looks like the soda clerk at the corner drug store. There is your chance. If you are going to act on the original statement it has got to be referred to some one you can picture, who looks like himself and not just like some one else. And, similarly, all definitions must refer back to the undefined.

Of course it does not help to play merry-go-round with your

definitions. If you define a plane in terms of lines, and a line in terms of points, and a point in terms of planes, you will not have undefined terms, be it said, but your definitions will be meaningless. If the soda clerk quit his job last week and departed for places unknown and you, not being addicted to sodas and cheap fiction, have never seen him; you may inquire about his looks. If the answer is that he looks like you, you will have got exactly nowhere in your investigation. You cannot define a thing significantly in terms of itself, no matter how large the circle, any more than you can moor a boat to one of its own cleats. All definitions have to be moored to the undefined.

The Game of Geometry: II. The Rules

But now, having our undefined term or terms, and having defined a whole series of more complex terms in relation to it or them, we must develop rules for the game of geometry. When Euclid worked out his famous system he set down a series of postulates and axioms which governed the game for centuries. He assumed, and most mathematicians followed his example, that these were the only rules according to which the game could be played.

Most of them are still accepted. There is probably no one who will question seriously the axiom that things which are equal to the same thing are equal to each other. But this type of rule is not good for all situations. It would not necessarily be true of friends, for example, that people who are friends of the same individual are friends of each other. When we are dealing in equal quantities, however, the relation will not be questioned. It is basic to both arithmetic and geometry. We should also agree that the whole is not greater than the sum of its parts. It may be true in literature that one stanza of a

poem is greater than the poem taken as a whole. There "greater" is used in a different sense. In arithmetic and geometry there is no question. These axioms are basic rules of the game. They are concepts common to all systems that deal in quantities.

But the story is different when it comes to the more specific rules of the game of geometry, the postulates. The fifth of these postulates in Euclid's version of the game has become famous: "If two straight lines in a plane meet another straight line in the plane so that the sum of the interior angles on the same side of the latter straight line is less than two right angles, then the two straight lines will meet on that side of the latter straight line."[3] This is known as the Parallel Postulate because on it depends the well-known theorem that through a point not on a given straight line there is only one parallel to the given line. There is good reason to believe that Euclid was more than a little embarrassed about this fifth postulate. He avoids using it until Theorem XXIX of Book I. But the postulate is essential to the system of geometry we all learned in school. It cannot be proven, any more than the other postulates can: it is a sheer assumption. But it is necessary to the system with which we are familiar.

What Happens to the System when the Rules Are Changed

It was not until the nineteenth century that it was fully realized, by Lobatchewsky, that another consistent system of geometry could be developed. Lobatchewsky substituted for the Parallel Postulate of Euclid another which allowed him to assume that an infinite number of parallel lines could be drawn through a point parallel to a line of which it was not a part.

[3] Young, *Fundamental Concepts of Algebra and Geometry*, Macmillan, p. 11.

Using this new postulate, he was able to work out a *non-euclidean* geometry in which a curved line is the shortest distance between two points, parallel lines meet at infinity, and the sum of the angles of a triangle is less than two right angles! So the ball started rolling. Soon there was another non-euclidean geometry, based on another fifth postulate, one that allowed the assumption that no parallel lines could be drawn through a point parallel to a line of which it was not a part. In this system there are no parallel lines and the sum of the angles of a triangle is more than two right angles.

And this is not all. There are many geometry games. There is one called "projective geometry," which deals only in the various intersectional properties of points, lines and planes. The system which we learned in school is, in contrast, based on the idea of measurement. What would you think of a geometry of four dimensions? Or one of five dimensions? These things are possible, too, in playing the game of spaces. You will be employing terms which have no familiar application, and the rules you will establish for moving around in four or five dimensional space will be highly imaginative, but you will have a system which is not only possible (from the game standpoint) but also consistent. There is also a geometry called non-archimedean, based on a rather unusual method of measuring angles between curved lines. The things you can do in selecting your terms and making rules for a geometry are both extraordinary and fascinating. Most of them do not describe a recognizable world. We do not, so far as we know, live in a world of five dimensions, nor in one in which any number of parallel lines can be drawn through a point parallel to a second line. But it is fun to play around with these eccentric space systems, and they illustrate well the power of the original rules in determining the character of a system.

There is a distinction often made between *pure* mathematics

and *applied* mathematics. The former enjoys fooling around with any mathematical system regardless of its application. We said earlier that you could make any system you wished out of numbers, whether or not you could count objects or make change with it. It is now apparent that you can do the same thing with points and lines and planes. If you are going to stick to geometry your terms will have to be points and lines and anything that can be defined in terms of them, but you are quite free in your choice of postulates. If you are going to play chess you must use the chess men, but you can play chess in two dimensions or three or four or five! A three dimensional chess board would look like a tenement house, eight glass boards on top of one another! In four dimensional chess you might take a time factor into account as the fourth dimension. If your opponent tried to capture your queen you could probably point out to him that while your queen *is* on her seventh square on Floor Six, it happens to be last week Tuesday in that square! If you want to devise rules for a five dimensional game —go ahead. I prefer to keep my sanity.

It is in this sense that geometry is like a game. You can take the basic terms of geometry and build them into any kind of a system that is interesting to explore. It is just like a game of solitaire. By choosing your rules as you wish you literally invent the world in which you are going to play: it may be a very dull world, it many have corners that are interesting, it may be so complex that it is almost impossible to deal effectively with it. Like number systems, some of the systems of geometry will describe the world in which we live and hence be useful as well as interesting. Most of us will concentrate on useful geometric systems. They are difficult enough. But by fooling around with systems that have no application to our world the geometry fiend has not only had a good time himself, but at the same time has thrown an important light on how deductive

systems are constructed and given us a good idea of the importance (and the arbitrariness!) of our axioms and postulates.

The Game of Geometry: III. The Combinations

The theorems that are developed in order in the geometry books are the later aspects of the system, the various outcomes of the initial postulates and axioms. In some ways developing a geometric system, or any deductive system, is like blowing up a balloon which has a picture of Mickey Mouse on it. You see Mickey in more and more clear detail as the balloon gets larger, but everything you see was there in the beginning. All of the possibilities of a geometrical system are there in the postulates and axioms. Change the postulates and you change the system. But given these rules of the game, you have already defined the system even though you have not worked it out in detail.

If you have a fertile imagination and want some fun, you might amuse yourself by making up your own list of rules for playing the game of geometry and see what kind of a "space" you get. We should tax the imagination too much if we tried to alter the statement of axioms: it would be very difficult to think of a space in which the whole is equal to only two-thirds of its parts taken together! Try it, if you do not see what I mean. But with the postulates it is different. Consider these possibilities:

1. It shall be possible to draw two straight lines joining any two points. One of these lines shall be twice the length of the other.
2. A straight line may not be more than six inches in length.

Add these postulates to the other five of Euclid and you will get a system of points and lines and planes such as no man

has ever experienced and, I am willing to wager, such as no man has ever thought of before. The entire system would have to be contained within a six inch sphere. Within the sphere every figure bounded by straight lines would become two figures, the second with sides twice the length of those of the first. And so on. It would be a curious system.

Two Requirements of Any Set of Postulates: Non-Contradiction and Independence

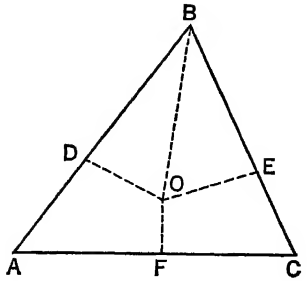
Can we take *any* set of postulates and make a system out of it? From the crazy thing we have just done above it might seem so. But there are limitations. Using the familiar postulates, Euclid was able to demonstrate that the interior angles of a triangle sum up to exactly two right angles. Suppose that, using the same counters and the same rules, he had also been able to prove that the sum of the interior angles of a triangle is equal to *four* right angles? Something would obviously be wrong. This would be like the situation presented by the famous proof in algebra that two equals four:

$$\text{Let } a = 2 \text{ and } b = 2$$

$$\begin{aligned}
\text{Then} \qquad a &= b \\
a^2 &= ab \\
a^2 - b^2 &= ab - b^2 \\
(a + b)(a - b) &= b(a - b) \\
a + b &= b \\
a + a &= a \\
2a &= a \\
4 &= 2
\end{aligned}$$

There is a similar situation in geometry, presented by the famous "proof" that any triangle is an isosceles triangle: [4]

[4] Adapted from the proof given on p. 48 of Ball's *Mathematical Recreations and Essays*, Macmillan, 1911.



Let ABC be any triangle.
To prove that AB = BC

Proof

Let OB bisect $\angle$ B
Draw perpendicular bisector of AC, meeting OB at O
Draw AO and OC
Draw perpendiculars from O to BC and AB

OB = OB	identity
$\angle$ DBO = $\angle$ EBO	construction
$\therefore$ $\triangle$ ODB = $\triangle$ BEO	two rt. triangles are equal if the hypothenuse and one adjacent angle are equal to the hypothenuse and adjacent angle of the other.
AO = OC	OF is perp. bisector of AC
OD = OE	$\triangle$ ODB = $\triangle$ BEO
$\therefore$ $\triangle$ ODA = $\triangle$ OEC	two rt. triangles are equal if the hypothenuse and one side of one are equal to the hypothenuse and one side of the other.
DB = BE	Homologous parts of equal triangles.
AD = EC	Homologous parts of equal triangles.
$\therefore$ AD + DB = BE + EC	If equals are added to equals the sums are equal.
or AB = BC	
	Q.E.D.

Obviously something is very wrong in these two cases. Can you find the errors? What has happened, of course, is that certain of the principles involved contradict others by which we know that four is not equal to two and that not every triangle is isosceles. If we accept these proofs as parts of our familiar systems then those systems harbor contradictions. And a contradictory system is clearly useless even as a game.

Remembering Mickey Mouse, we realize that if the rules of our system are contradictory then the result will be contradictory theorems. To be confronted with contradictory theorems in one and the same system is, to put it mildly, embarrassing. It is just like the case of the game of checkers in Moscow. Two people cannot play the same game of checkers according to different rules. My friend was playing according to a rule of the game which states that men can move backward only after they have reached the eighth row. His Russian opponent was observing a rule to the effect that men can be moved forward or backward at any time. Obviously these two rules contradict one another. The result can only be chaos. You would be at a loss to know how to play a game in which the rules contradicted one another, as would be the case, for example, if you found both my friend's rule and his opponent's in the same rule book describing the same game. And, similarly, it is impossible to work out a system of arithmetic or geometry if the initial postulates are contradictory, for if they are (remember Mickey Mouse!) they will surely be responsible for contradictory theorems later on. This is just as certain as it is that you and your checkers opponent will get into a terrible fight sooner or later if you try to play with conflicting rules. Hence one condition which a deductive system must fulfil is that *the postulates be consistent.*

Just as important from the point of view of indicating the fundamental structure of the system is a second rule which a set

of postulates should obey. If you are inventing a game you will want to set down rules to cover every possible situation, but at the same time you will want to avoid cluttering your rule book with too many regulations. You would be at fault, for example, if you failed to state explicitly whether a player had to capture an opposing piece when he was in a position to do so or could exercise choice in the matter. It makes a real difference to the game. But suppose you are playing on a checkerboard and that in trying to make your rules complete you set down the two following:

> Rule 3: Pieces move only diagonally.
> Rule 4: Pieces must remain on squares of the same color.

Now, in your anxiety to be explicit in your instructions, you have overstepped the mark, for Rule 4 follows necessarily from Rule 3. And, similarly, in geometry you will want to have *sufficient* postulates so that the system can be worked out in all of its details. But at the same time the postulates must be *independent* of one another. It would be wasteful to put down a postulate which could be proven by using the others. For example, in my geometry book Theorem V states: "From a point without a straight line one perpendicular, and only one, can be drawn to this line." This statement is so elementary that the constructor of a geometrical system might be tempted to place it among the postulates. But the fact is that it is unnecessary to do so, because it can be proven on the basis of the postulates already set down. This particular theorem is, in my book, based upon the postulate that two straight lines which have two points in common coincide throughout their whole extent and form but one line.

Though it goes too far afield to be discussed at length here, any one interested in the construction of deductive systems will enjoy investigating the problem of which statements should

be chosen as postulates of a system, and which appear later as theorems. Geometricians have amused themselves by building the same geometrical system out of different sets of postulates and, as one would expect, some like one set of postulates better, some another. There is no reason, of course, why Theorem V should be a theorem and not a postulate. If you increase its importance by raising it to the rank of a postulate you will find that the "postulate" on which it was founded in the days when it was a mere theorem may now become a theorem itself. Take your choice. The only question you will want to ask is, "Which of these statements is more fundamental to the system I am trying to develop?" Use the more fundamental one as a postulate and prove the other in terms of it.

From the Logic of Mathematics to the Logic of Logic

We have seen that all deductive systems are games played with definite sets of men and according to given sets of rules. Some are recreational, some are developed out of sheer curiosity, and others have application to life. But all are constructed in the same way and all are mental in origin. All use elements which are manipulated on a certain field according to definite rules. Within these systems we develop judgments which can be proven beyond the shadow of any doubt. Change the rules and you change the system.

The mathematician seems to have a corner on deductive systems. Of all of the systems invented by human reason, arithmetic and geometry are among the most useful and most familiar. But there are many others: algebra, trigonometry, and calculus being among them. It is not difficult to understand why most mathematicians are good at games of skill. Dealing in structures all of the time, they appreciate them

more than most. The field of logic can boast two equally important and interesting deductive systems, ones which deal with logical counters and which employ the various logical relationships that we have found to hold between logical terms. To these systems we shall now turn.

Chapter Ten

THE GAME OF LOGICAL PROPOSITIONS

WHEN people start talking about their favorite books the old question of what ten books they would take to a desert island inevitably arises. The question as to which are the ten most important books is still another. If you were asked to name ten books written in the twentieth century that will surely be read two thousand years from now, if man does not completely destroy himself before he learns how to play with his new toys, what books would you name? The chances are large that your list of favorite books and of important books does not coincide in a single item, a fact worth thinking about by the way. I should not like to have to list ten books that will surely be read in two thousand years, but I could easily name one. It is a book which few have heard about, which many less have dipped into, and which only a handful of men have read from cover to cover. It would take high honors in the class of Books-Difficult-to-Read and highest honors in the class of Books-Difficult-to-Write. The work which it represents is breath-taking: its importance to an understanding of human thinking is inestimable. It is *Principia Mathematica*, by Messrs. Whitehead and Russell. But I should certainly not want to be cast up on a South Sea island with it as my Man Friday.

If you have the courage to open this book you will find in it the details of a deductive system which belongs exclusively

to the logician, the system of propositions, sometimes called The Calculus of Propositions. It is as much a game as any of the other systems we have examined, but like geometry it is a game which has an important application to human life. Geometry shows us, with systematic care and with an eye to fundamentals, the various relations which points and lines and planes have to each other. The system of propositions endeavors to show in the same manner the ways in which the propositions which make up man's thinking are interrelated. In its technique it is a constant reminder of the more familiar geometry. But it is so basic to the problem of finding the structure of thinking that it holds a place in the field of logic similar to that of arithmetic in the field of mathematics. All mathematics goes back to arithmetic and, as we shall later observe, all logic goes back to the system of propositions.

Unfortunately the road ahead is rough. And, even more unfortunately, it is through a terrain where detours are not possible. The system of propositions is complex. Conscious effort and persistent study are necessary to an understanding of it. Small details are often highly important. Each step is carefully planned and must be executed with precision. I know already how you are going to feel about all this at first: I have the same feeling when an economist tries to explain the intricacies of foreign exchange. There are some things in this world which by their very nature are difficult to comprehend—like the first childhood experience in the dentist chair. The system of propositions is one of them. If you find this chapter strenuous, and perhaps painful, it is because it is unavoidably so. The logician is in a predicament—he cannot use an anesthetic. You will want to read these pages carefully and slowly. If you find that you are losing your grip, stop awhile and then come back for a fresh attack.

The Necessity of Proving the Validity of Logic Structures

It may not have occurred to you at the time, in fact your reaction was probably quite the opposite, but when we were talking about the structure of implicative and disjunctive arguments we were surprisingly superficial. We took a lot for granted.

If some fiendish geometrician should take you by the lapel and ask whether you thought that two triangles are equal if the three sides of the one are equal respectively to the three sides of the other, you would certainly agree that they are. You would see no reason to doubt the statement. When I buttonholed you in Chapter Four and asked you to agree that when one proposition implies another and the first is accepted as true, then the second must also be accepted as true, you did not object either. At an early age you realized that when mother said that if you ate too much ice cream you would be sick, and if you then proceeded to eat too much ice cream, an uncomfortable experience *was* ahead. This, like the proposition about the two triangles, was just common sense. But what about this thing called "common sense"?

The fiendish geometrician tugs at your lapel again and says: "*Why* is this statement about triangles true?" You may reply weakly: "I don't know. It just is." But you are getting nowhere fast. The geometrician wants a *proof* of his proposition. Indeed if you will open my edition of his geometry you will find that this statement about the two equal triangles is asserted as Theorem XXXIII of Book I, is carefully demonstrated, a reason given for each step, and the famous Q.E.D. printed at the end of the demonstration. That is how careful the geometrician must be. And the logician must be equally careful. How do we know that this structure is valid?

$$\begin{array}{l} P \supset Q \\ P \\ \hline \therefore Q \end{array}$$

In Chapter Four we simply assumed that it was without seeking to prove it. It worked successfully in the examples we cited, so we let the matter go at that. But this is conduct unbefitting a logician.

We are now to discover that the implicative argument is but a small part of a much larger structure, one developed as systematically and clearly as geometry, one with its own axioms and postulates, one whose fundamentals can be studied as carefully as men have studied the fundamentals of Euclid's system. Since it is a system *of propositions* we shall be correct in suspecting that it is a structure given by a *molecular* analysis of human thinking. Propositions are studied as wholes, and, on the basis of certain elementary rules of the game, the relationships among whole propositions worked out. And we shall not be surprised to learn that when we work out the system we are in a position *to prove* the validity of the structure of the implicative argument. So now to our task.

The Game of Propositions: Undefined Terms

The counters with which we will play this game are propositions. We shall symbolize them with the letters of the alphabet beginning with P (for Proposition); P, Q, R, and so on. We shall assume that every one knows what a proposition is, and hence not try to define it. Our glimpse of the structure of geometry showed us that in any system there must be some term or terms undefined. A beginning must be made somewhere. So our beginning will be with the molecule of logic, the proposition. When P (or Q, or R, etc.) appears alone we

shall read it as "P is true," whatever proposition P symbolizes shall be taken to be a true proposition.

We shall also find it handy to be able to say, "P is false." We shall indicate this by writing –P. The expression can be translated in various ways into words: "P is false," "Not–P is true," "It is not true that P is true." This idea of "not" shall also be undefined, on the ground that it is known what we mean when we speak of the falsity of a proposition. It is as if we might say at this point that we have decided to play a game with counters which are exactly alike (in being propositions) except that some of them will be white (true) and others will be black (false). Since this game is being played with ideas and not physical objects, we shall have to use symbols instead of paint to indicate the difference.

One more undefined idea is needed before we can begin to build the system. We must have some way of joining these "black" and "white" propositions. If we had gathered together all of the numbers in arithmetic and were about to play with them, we should need some way of putting them together, some kind of cement for the system. In arithmetic the "cement" is the idea of *addition*, symbolized by +; and, being fundamental to the system, it is undefined. The fundamental relation in our system of propositions we shall take to be the idea of disjunction, symbolized by the now familiar v. And when we write "P v Q" we shall read it as, "Either P is true or Q is true and perhaps both are true." It is the logical equivalent of the legal "and/or," to which we referred earlier.

These three ideas are simple enough. But why do we speak of them as undefined? It cannot be denied that we have taken words from ordinary language and borrowed from them meanings for these symbols. To speak more accurately we must say that they are undefined *within the system*, they are the ideas with which the system starts. They are to the system of propo-

sitions what primitive words are to our modern vocabularies. We have seen that most of our words are defined in terms of others but that somewhere if we are not just to go in circles we must point at objects and say, "This we shall call a 'man,'" "This we shall call 'fire.'" Some logicians prefer to call these three ideas *primitive ideas*, because they must be accepted in the primitive stages of the development of the system and *all other ideas used in the system must be defined in terms of them.* Everything in the system refers back to them.

The Game of Propositions: A Defined Term

We should not get far in conversation if every time we used a new word we had to point to the object for which it stands or describe by gesticulations the idea intended. So we use definitions, and increase our vocabularies. We should not get far in arithmetic if we could not use the idea of multiplication, and its symbol, $\times$. If instead of writing 6×8 we had to put down $8 + 8 + 8 + 8 + 8 + 8$ our arithmetic would soon be very cumbersome. In geometry, if we had to use "a figure bounded by three straight lines" every time we meant "triangle" we should be in the same trouble. The symbol $\times$ is a shorthand for a complex series of additions: "triangle" is shorthand for a certain type of geometrical figure. In other words, *multiplication* is defined in terms of *addition*, and *triangle* is defined in terms of *line*. Each adds a new idea to its system. We get so accustomed to these new words and ideas that we do not need to refer back to the definitions every time we employ them. They add greatly to our convenience in using the system. *But they are not primitive ideas: they are ideas defined within the system.*

In the system of propositions certain defined ideas have been

found very useful. For example, if we take our three primitive ideas and put them together in this fashion:

$$-P \vee Q$$

we get an interesting combination which under another guise would be quite familiar to you because you use it often in dealing with propositions. Do you recognize it? Probably not. Well, let us look at it carefully and in detail. As it stands it should be read, "Either P is false or Q is true and perhaps both." We are faced with this situation:

i. if P is false, Q may be either true or false.
ii. if Q is true, P may be either true or false.
But iii. if P is true (i.e. not false) then Q is true,
and iv. if Q is false, then P is false.

Now do you recognize the combination? How do we speak of two propositions when they are related in this way? If the first is true, the second must be true: if the second is false the first must be false. We say that the first proposition "implies" the second. Remember the episode of mother and the ice cream?

This is an important discovery. We suddenly find ourselves in a position to *define* the relation of implication among propositions in terms of primitive ideas. In accordance with our custom in an earlier chapter we shall use the "horseshoe" to symbolize the relation. And, remembering the function of a definition, we shall be justified in using an equal sign to indicate that we are engaged in defining an idea new to the system. Thus:

$$(P \supset Q) = (-P \vee Q) \qquad \textit{Definition A}$$

This means that the two expressions on either side of the equality sign equal one another in the sense that they are inter-

changeable in the system: one can be written for the other at any time.

As the system of propositions advances in complexity the use of parentheses is found to be awkward and cumbersome. To meet these difficulties the logician has developed a shorthand of dots to take the place of parentheses. Using the dots notation, the definition of implication would be written:

$$P \supset Q \,.\!=\!.\, -P \vee Q \qquad \textit{Definition A}$$

The dots serve here to set off the equals sign in a way that will indicate its relative importance in the expression.

Practice in Reading the Symbolism of This New System

These dots serve a vital function in the system of propositions. Unless they are thoroughly understood and accurately read one may become hopelessly confused.

It is their function to make clear what would otherwise be ambiguous. This can be illustrated by citing a similar threat of ambiguity in arithmetic. If I write down this expression:

$$3 + 4 \times 5$$

do I mean $3 + (4 \times 5)$?
or $(3 + 4) \times 5$?

It makes a difference. The result in one case is twenty-three, and in the other case, thirty-five. We had to make some rule to cover the ambiguity. As you know, the rule is that the multiplication shall be performed before the addition.

Turning now to propositions, we are faced with a similar threat of ambiguity. If I present you with this expression:

$$P \vee Q \supset R$$

do I mean $P \vee (Q \supset R)$?
or $(P \vee Q) \supset R$?

Again, it makes a difference. In this case, however, we have no rule. Disjunction and implication are of equal importance. Hence we employ dots to indicate what is intended:

$P \vee (Q \supset R)$ is written $P . \vee . Q \supset R$
$(P \vee Q) \supset R$ is written $P \vee Q . \supset . R$

In each case the dots are placed either side of the more important relationship.

We shall often find ourselves confronted with situations more complex than this, situations in which there are relations of more than two orders of importance. In such cases the logician simply increases the number of dots on either side of the most important relation. One of the theorems we shall encounter, for example, asserts that if P implies that Q implies R, then Q implies that P implies R. Using parentheses this would be written:

$$(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$$

But the logician puts it more simply thus:

$$P . \supset . Q \supset R : \supset : Q . \supset . P \supset R$$

Theorems are not always as well balanced as this. Sometimes the double dots are needed only on one side of the most important relationship:

$$(Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$$

is written

$$Q \supset R . \supset : P \supset Q . \supset . P \supset R$$

And occasionally in proofs of theorems we encounter situations so complex that groups of three dots are necessary to indicate four orders of importance in relationships. Thus:

$$P \vee P \,.\supset.\, P :\supset:.\, P \,.\supset.\, P \vee P :\supset.\, P \supset P$$

With a little practice you will easily read such an expression. The position of the three dots together indicates that the second implication sign from the left is the most powerful relationship in the entire expression. Perhaps you are suspicious of this whole idea of using dots. But surely you are misled by their unfamiliarity. Try to use parentheses on that last expression and see what a mess it becomes:

$$((P \vee P) \supset P) \supset ((P \supset (P \vee P)) \supset (P \supset P))$$

Believe it or not, the dots *are* a useful shorthand.

Some Possible Defined Terms in Arithmetic and in the System of Propositions

Let us pause for breath at this point. We started out with three primitive ideas (proposition, falsity, and disjunction) and to them we added, by means of a definition, another idea (implication). One of the most interesting things about this, and any other deductive system, is that the choice of which ideas shall be primitive and which defined is purely a matter of convenience. We saw, for example, that in geometry "circle" could be primitive and "point" defined in terms of it, or *vice versa*. We might have selected *implication* as a primitive idea in the system and defined *disjunction* in terms of it. Can you figure out what the definition would be in that case?

It will be helpful to bear in mind that our task here is something like the task of symbolizing arithmetic ideas. We want to develop symbols that will be useful and significant.

In both cases the system was in use before its structural character as such was studied. Hence in both cases the ideas which are defined for use in the system will follow closely ideas which are already familiar. In arithmetic the idea of multiplication was very useful, and could easily be defined in terms of addition. But there are countless other ideas which *might have been* defined. The arithmetician might have established a relation between numbers, symbolized by ♯ (call it what you like!), and defined it as:

$$a \sharp b \,.=.\, (a \times b) + a + b \qquad \textit{Definition}$$

$3 \sharp 5$ would be 23. The reason why he does not bother with this relation is that it is not of any great importance to the system, and there is no point in making the system more complex than it needs to be. This is the law of parsimony coming in again. If the system can be expressed conveniently and adequately without defining a new and unfamiliar term, then that term should be omitted.

Suppose we try the same kind of experiment in the system of propositions. Suppose we put our primitive ideas together in a new way and see what we get. Take the expression:[1]

$$-(P \vee Q)$$

It is false that either P is true or Q is true. This time we have not gone so far afield, for you will immediately recognize this as the familiar "neither . . . nor" relationship between propositions. Most systems avoid complexity by leaving this out, but in others it is used as *the* primitive relationship!

Let us try again for a combination. Suppose we write down:

$$P \mathbin{?} Q \,.=.\, -(P \vee -Q) \qquad \textit{Definition}$$

[1] Parentheses *are* used in cases like this, to indicate that a negation covers an entire expression.

The new symbol (?) indicates a relationship between any two propositions such that it is false that either the first is true or the second is false. In other words, the first is false and the second is true. That relationship is so unfamiliar that we have no name for it. It does not coincide with any relationship between propositions which is familiar. We could perfectly well incorporate this definition into our system and use the symbol. But there would be as little point in doing so as in using ♯ in arithmetic. It is obviously undesirable to clutter up a system with unnecessary details.

This will illustrate two points of importance to an understanding of deductive systems. The first is that the selection of symbols is quite arbitrary. We could take *any* group of symbols as a starting point, work out *any* defined terms, and develop a system *without once assigning a meaning to either the notation or to the propositions worked out from it.* The resulting system would have structure, but no significance as anything but a game. Suppose at the beginning of this chapter we had introduced P, Q, –, and v to you as nothing but black marks on white paper, then proceeded to define another black mark (horseshoe) in terms of them. We could have devised an entire deductive system using these symbols, worked it out as a sort of game, without attaching significance to the symbols. And then, when it was all complete, we might have discovered that *if* you call P a proposition, v a disjunctive relation, and so on, the system we had devised could be interpreted as a system of propositions. This would, of course, have been an extraordinary coincidence. How would we have proceeded to make up rules for the conduct of a set of symbols to which we attached no meanings? We could only have done so blindly. And the chances that we should have hit upon the rules of conduct of propositions are infinitesimal. Finding a needle in a haystack would be simple in comparison. No deductive systems except

pure games are ever developed in this way. In all other cases we have a useful result in view.

The Game of Propositions: The Rules

So now to proceed toward the useful result of establishing a system of propositions. We have the counters for our system. What shall be its rules? We need rules which shall correspond to the axioms in geometry. Knowing that we are developing a system *of propositions* we have some guide as to what rules would be pertinent. It would be foolish, for example, to write:

$$\text{Rule:}\quad P \supset Q$$

This would assert that any proposition implies any other. "It is raining" would imply "Japan is waging an undeclared war." In our experience with propositions we do not find this to be so. We must be more careful. We must look for rules, governing relationships among propositions, which accurately describe the actual habits and customs of propositions. Propositions do have their little eccentricities.

Furthermore these rules will have to be so obvious as to go unquestioned, rules as obvious as the axioms of geometry. When I set down the ones generally accepted you will think them extremely silly. Every neophyte does. But I ask you to remember your school-days' experience with the axioms of geometry. They seemed so obvious that you laughed at them. But remember what you were later able to do with points and lines and planes because of them! Remember the thrill of the later theorems.

There are five rules of the game of propositions. They are sometimes called "primitive theorems" because they are assumed to be true and no proofs of them are given.

Rule I. $P \vee P . \supset . P$

This is called the Principle of Tautology. If either Professor Hoppelpoppel is a good teacher or he is a good teacher, then he is a good teacher. You may think this a sheer waste of time, but you will not question the truth of the rule!

Rule II. $P . \supset . P \vee Q$

This is the Principle of Addition. If Professor Hoppelpoppel is a good teacher, then either he is a good teacher or German is a popular subject. Do not laugh too loudly: you will see what we are doing in a moment.

Rule III. $P \vee Q . \supset . Q \vee P$

This is called the Principle of Permutation, and tells us that disjunction is a symmetrical relationship. Either Professor H. is a good teacher or German is a popular subject implies that either German is a popular subject or Prof. H. is a good teacher.

Hold your horses now! Suppose you put an implication in place of the disjunction and made the rule state that if P implies Q then Q implies P. Then you *would* be in trouble. Implication is not a symmetrical relationship. There is more in these rules than meets the eye.

Rule IV. $P . \vee . Q \vee R : \supset : Q . \vee . P \vee R$

This is the associative principle. If Prof. H. is a good teacher or either German is a popular course or German classes are large, then German is a popular course or either Prof. H. is a good teacher or German classes are large.

You have stopped smiling? This assumption is not as obvious as the others. You will have to think a minute to appreciate what it says. It definitely describes a property of propositions. It would not, for example, hold for men and their relative visibility. You would not say that if P sees that Q sees R, then Q sees that P sees R.

Rule V. $Q \supset R . \supset : P v Q . \supset . P v R$

This Principle of Summation is even more complex. If German is a popular course implies that German classes are large, then either Prof. H. is a good teacher or German is a popular course implies that either Prof. H. is a good teacher or German classes are large. You will have no difficulty in accepting this assumption if you think about it a moment.

Though it is difficult to believe, all of the properties of propositions are wrapped up in these five rules. The problem now is to bring them to expression.

Learning to Play the Game

One's first attempts to play a new game are always awkward. The counters are not familiar yet, and there are many hasty consultations of the Rule Book. Though the game of propositions employs at first only four counters and five rules, both are so unfamiliar that one has constantly to refer back to them in making the first moves. Fortunately they are so few that we can easily commit them to memory, or scribble them on a cuff or an index card, so that they will continuously be before us. The properties of the game in a nut-shell:

Counters.	Undefined:	P, Q, R	propositions	
		–	is false	
		v	either . . . or	
	Defined:	$\supset$	if . . . then	*Definition A*

Rules.	I	$P v P . \supset . P$
	II	$P . \supset . P v Q$
	III	$P v Q . \supset . Q v P$
	IV	$P . v . Q v R : \supset : Q . v . P v R$
	V	$Q \supset R . \supset : P v Q . \supset . P v R$

You will find the game much easier if you will take the trouble to make these things so familiar that you can rattle them off as you would the alphabet or the multiplication table. Eat them for breakfast, play tennis with them, dream about them—do anything that will make them so much a part of you that you could recite them standing on your head. You will be rewarded.

How a Theorem Is Proved

When a magician pulls a rabbit out of a silk hat we are amazed and wonder what is the secret of the trick. When a geometrician takes out of his hat the theorem that two triangles are equal if the three sides of the one are equal respectively to the three sides of the other we are not at all astonished, and very few of use bother to stick around after the show to see how it is done. The logician has the same difficulty in captivating his audience. A good many of the "rabbits" produced in the system of propositions are old friends and we accept their appearance without surprise. But there is a trick to the thing nevertheless, and if you are going to understand propositions and their interrelations you must learn the trick just as carefully as you must learn the demonstrations of the various theorems in geometry if you want to understand the properties of points and lines and planes.

Every child is aware that if he gets a good spanking he will eat his next meal standing up. Let us suppose that his mother, who is efficient with the hairbrush, tells him that if he squirts her once more with his water pistol he will get a thorough spanking. It does not take much cogitation on the part of the child to come to the realization that in this case if he squirts her just once more he will take his next meal standing. The conclusion follows as the night the day. Put in more general language, and in terms of propositions, this is to say that if

one proposition implies a second, then if a third proposition implies the first that third also implies the second. Symbolically it is even more clear:

$$q \supset r . \supset : p \supset q . \supset . p \supset r$$

If q implies r, then if p implies q it also implies r. This is just common sense in the realm of propositions, a recognition that "implies" is a transitive relation. It is like saying that if A is taller than B, then if C is taller than A he is also taller than B. But common sense is not enough in logic any more than it is in geometry. Every step forward must have a Q.E.D. attached to it.

As a matter of fact this statement about p and q and r is one of the first theorems proven in the *Principia Mathematica*.[2] In order to acquaint ourselves with the technique of this system, let us examine its proof with care:

Theorem 1

To prove: $q \supset r . \supset : p \supset q . \supset . p \supset r$

Referring to Rule V, which we assume to be true and fundamental to the system, we assert for any three propositions, p, q, and r:

$$q \supset r . \supset : p \vee q . \supset . p \vee r \qquad \text{(i)}$$

Then, since p stands for *any* proposition, we may substitute $-p$ for it, and we get

$$q \supset r . \supset : -p \vee q . \supset . -p \vee r \qquad \text{(ii)}$$

But looking back at the definition of implication (A) we see that for *any* two propositions $P \supset Q$ is identical with $-P \vee Q$. Hence we may say that

$$p \supset q . = . -p \vee q$$
$$\text{and } p \supset r . = . -p \vee r$$

[2] The order and numbering of these theorems are those of the present author.

And since the two sides of each of these equations are identical (i.e. the same thing stated in different symbols), we may substitute the expression on the left wherever the expressions on the right occur. Now both of the expressions on the right appear in (ii). Substituting the expressions on the left for them, we get:

$$q \supset r . \supset : p \supset q . \supset . p \supset r$$

Q.E.D.

When put in more concise form this proof might be written:

$q \supset r . \supset : p \vee q . \supset . p \vee r$ (i) by V
$q \supset r . \supset : -p \vee q . \supset . -p \vee r$ (ii) subst. $-p$ for p in (i)
$q \supset r . \supset : p \supset q . \supset . p \supset r$ (iii) by A

Q.E.D.

You are probably wondering why in proving theorems we suddenly switch from capital letters to small ones. Most statements of this system use letters of the same size throughout. Perhaps our innovation is a mistake, but I think not. One of the difficulties frequently encountered in understanding this system as customarily written is due to the mistaken idea that the P's and Q's and R's of the definitions and postulates are the P's and Q's and R's of the later propositions, respectively. A little thought tells us that since P and Q and R are *any* propositions the definitions and postulates could employ any of them. For example, the definition of implication could be written:

$$Q \supset R . = . -Q \vee R$$

or $$R \supset P . = . -R \vee P$$

It makes no difference. Seeing the definition always in terms of P and Q suggests the erroneous idea that these are the only two letters that can enter into the definition, and further misleads us into assuming that when we speak later of P and Q

and R we are referring to identically the same propositions employed in the definitions and postulates.

Since P and Q and R symbolize *any* propositions, the situation is clarified by changing slightly the symbols when we come to the theorems. We must take care only that whenever the same letter appears more than once *in a single assertion* it refers to the same proposition *throughout that assertion.* Wherever Q occurs in Rule V, for example, it always refers to the same proposition. But as between Rules IV and V the two Q's may stand for entirely different propositions.

We have other old friends among the early theorems in the *Principia.* You are probably tired of hearing about the boy who ate too much ice cream. To relieve the monotony let us make a change in the wording, present the same argument in slightly different form. "If you eat too much ice cream you will not be well. Hence if you are well, you did not eat too much ice cream." Giving letters to the three different propositions involved here, we get:

$$p \supset -q \,.\supset.\, q \supset -p$$

This we shall call Theorem 2. Its proof is as follows:

Theorem 2

$p \vee q \,.\supset.\, q \vee p$	(i)	Rule III
$-p \vee -q \,.\supset.\, -q \vee -p$	(ii)	subst. $-p$ for p and $-q$ for q in (i)
$p \supset -q \,.\supset.\, q \supset -p$		by A

Q.E.D.

Here is another theorem with which we already had a speaking acquaintance, but which we now know more intimately because we are able to prove it in terms of the system of which it is a part.

One of the recognized techniques in a debate is that of reduc-

ing your opponent's contention to an absurdity. Suppose you are arguing the old question about the relative merits of the victories of war and of peace. Your opponent argues that we gain what we are seeking, "security" or "a world safe for democracy" or what not, by fighting wars and winning them. You may then meet him on his own ground. "All right. Go out and fight wars and win them. Where does it get you?" And you proceed to show that to win a war is to lose it, to give up the things for which you have been fighting. If you carry this technique through successfully you will have an impregnable refutation of the other fellow's argument, for we all recognize that if a proposition implies its own falsity it must be false. This too can be proven:

Theorem 3

To prove: $p \supset -p . \supset . -p$

$p \vee p . \supset . p$	(i)	by Rule I
$-p \vee -p . \supset . -p$	(ii)	subst. $-p$ for p in (i)
$p \supset -p . \supset . -p$		by A

Q.E.D.

If a proposition implies its own falsity we know that something is wrong.

The situation is like that in the medieval chestnut about the town officials who stopped all who entered the gates of the city and asked them where they were going. If they told the truth they were allowed to proceed: if they lied they were hung at the gallows close by. One day a famous outlaw tried to enter the city. When asked where he was going he said: "I am going to be hung at that gallows." What should they do with him? The truth of his proposition certainly implied its own falsity: if he told the truth he would have to be hung in order to make him truthful, but the reward for telling the truth was freedom. On

the other hand, if he told a lie he would have to be freed in order to make him a liar, but the punishment for lying was to be hung!

Modern democracies are faced with a similar situation as regards the free speech issue. They are asked to grant free speech to a group which uses the freedom to advocate censorship! Democracy is asked to respect a civic right even when used to destroy that right. There is something self-contradictory in absolutely unlimited freedom of speech.

A More Difficult Proof

We remember that in our experience with geometry we soon encountered theorems which could not be called obvious. Taking the theorems of geometry in order we should expect them to get more complex and more difficult. And we must take them in order, because the proofs of the later theorems are dependent on the proofs of earlier ones. We saw, for example, that the proof of the theorem that vertical angles are equal referred back to the earlier theorem which stated that if two adjacent angles have their exterior sides in a straight line, these angles are supplements of each other.

If we add one more theorem to the three already stated in this new system, omitting the proof this time (you can find it in the *Principia* if you are interested), we open up a whole new set of possibilities.

Theorem 4

$$p . \supset . q \supset r : \supset : q . \supset . p \supset r$$

You will probably want to puzzle over this one. It says that if the truth of one proposition implies that a second implies a third, then the truth of the second implies that the first implies the third. Make up your own concrete application of this theo-

rem. If it has recognizable meaning you will soon see that there is no reason to doubt the theorem. You will also recognize that we are getting into a realm of complexity in which we are discovering relations among propositions which would never have been brought to light if we had not taken the trouble to study the system as a system. Similarly, there are many things which Euclid was able to show us about lines and points and planes which we should have missed entirely if the system of geometry had not been worked out carefully and logically. Deductive systems are powerful instruments.

We had to write down Theorem 4 because the proof of the next theorem, the 5th, depends upon it. And since the proof of Theorem 5 introduces a new technique it will pay us to examine it with care:

Theorem 5

To prove: $p \supset q . \supset : q \supset r . \supset . p \supset r$

$p . \supset q \supset r : \supset : q . \supset . p \supset r$ (i) by Theorem 4

This time we are going to make a more complex set of substitutions. p stands for *any* proposition, as we know. Instead of replacing it by another simple proposition, let us replace it by a compound proposition made up of two simple ones. If q and r are propositions, then $q \supset r$ is a proposition. Concretely, if *It is cloudy* and *It may rain* are propositions, then *If it is cloudy then it may rain* is a proposition.

Hence since the letters in Theorem 4 refer to any three propositions, we may substitute:

$q \supset r$ for p
$p \supset q$ for q
$p \supset r$ for r

So we proceed:

$$q \supset r . \supset : p \supset q . \supset . p \supset r . : \supset : . p \supset q . \supset : q \supset r . \supset . p \supset r \quad \text{(ii)}$$

subst. $q \supset r$ for p, $p \supset q$ for q, and $p \supset r$ for r in (i)

Now comes the new principle. (ii) says that

if $(q \supset r \,.\, \supset : p \supset p \,.\, \supset .\, p \supset r)$ is true, *then*
$(p \supset q \,.\, \supset : q \supset r \,.\, \supset .\, p \supset r)$ is also true

But it happens that in Theorem 1 we have already asserted that the first of these statements *is* true. If you will examine them carefully you will see that the first statement is exactly like the statement of Theorem 1.

We have proved Theorem 1. We know it to be true. Hence it follows that the second statement is also true, and can be asserted by itself:

$$p \supset q \,.\, \supset : q \supset r \,.\, \supset .\, p \supset r$$

Q.E.D.

This is not just sleight of hand. It is sound reasoning. But it is certainly correct to say that these proofs are more ingenious than those of geometry. The whole trick of devising a proof of a new theorem in cases like this is that of seeing exactly what substitutions should be made in order that the rules and theorems already at our disposal can be worked into the desired form.

Sometimes the substitution is obvious as, for example, in the case of Theorem 6, which asserts:

Theorem 6

$$p \,.\, \supset .\, p \vee p$$

Just a glance at the rules of the game shows that the simple substitution of p for q in II gives the desired result. Indeed Theorem 6 may be regarded as a special case of Rule II. But in more difficult cases, as that of Theorem 5, one needs a logician's intuition to know what to do.

An Obvious Theorem but a Difficult Proof

It does not always follow that if a theorem is easily stated the proof of that theorem is easily devised. Indeed just the reverse is the case when we come to Theorem 7. This is one of those theorems which one suspects might have been used as one of the rules of the system. On the very face of it no one would doubt it. But not being a rule it must have its proof, like all of the other theorems.

Theorem 7 simply states that any proposition implies its own truth, $p \supset p$. Its proof is interesting not only because it is an excellent illustration of the working of the logician's intuition but also because it twice makes use of the principle of proof introduced in working out Theorem 5:

Theorem 7

To prove: $p \supset p$

$q \supset r . \supset : p \supset q . \supset . p \supset r$ (i) by Theorem 1

$p \vee p . \supset . p : \supset : . p . \supset . p \vee p : \supset . p \supset p$ (ii)

subst. $p \vee p$ for q and p for r in (i)

But $p \vee p . \supset . p$ by Rule I

$\therefore p . \supset . p \vee p : \supset . p \supset p$ (iii)

But $p . \supset . p \vee p$ by Theorem 6

$\therefore p \supset p$

Q.E.D.

Careful study of this and the earlier proofs will be rewarding. They show how cautious we must be in taking each new step no matter how obvious that step may seem. Having given enough proofs to illustrate the technique we shall now take mercy on the reader and state the later theorems without their proofs. Any one who is a glutton for work and wants to see how good his logical intuition is may try to work out some of the later proofs for himself. We shall give hints from time to time

which will be helpful to this admirable endeavor on the part of the ambitious. But we ourselves beg off. After all, we are not rewriting the *Principia!*

Showing How Later Proofs Depend on Earlier Ones

It probably has crossed your mind that in proving Theorem 7 we kicked up a lot of dust without much result. This judgment is slightly premature. As a matter of fact Theorem 7 is one of the key theorems in the system. Moving more rapidly by omitting proofs, we shall get a new perspective which will show us the interconnections among theorems. Taking them in the order of their proof we get a series which builds on itself the way children's blocks do in erecting a tower. Take out the bottom block and the tower falls.

In the series of three theorems given below all of them will be recognized as important ones, and they all refer back directly or indirectly to Theorem 7.

Theorem 8. $-p \vee p$

The proof of this theorem employs Theorem 7. Can you work it out? The principle which it asserts is more familiar as *The Principle of Excluded Middle.* Any proposition is either true or false, there is no middle ground. Either he stole the money or he did not steal it. Either she will not marry me or she will marry me. It would be a queer world if this rule did not hold!

Theorem 9. $p \vee -p$

The Principle of Excluded Middle in another, the more usual form. In proving this the *Principia* employs Rule III and Theorem 8. With this hint can you work out the proof yourself?

Theorem 10. $p \supset -(-p)$

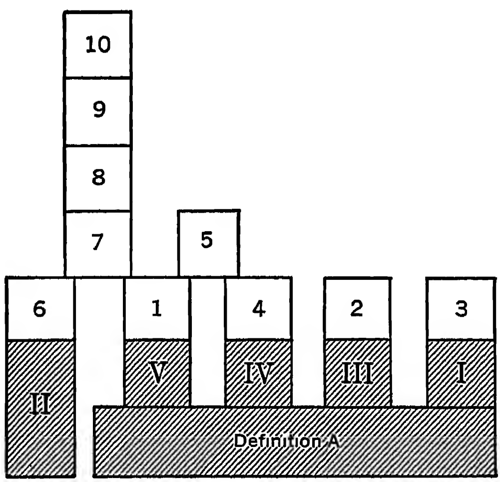
You will recognize this as the *Principle of the Double Negative,* a statement of the fact that in dealing with propositions two negations cancel one another. It is similar to the algebraic

fact that two minuses make a plus. If it is true that elephants have trunks, then it is false that elephants do not have trunks.

By making a substitution in Theorem 9 and then applying Definition A you can produce a demonstration of this theorem.

Now, I hope, you see the importance of that "obvious" Theorem 7. Theorem 10 is dependent on Theorem 9; Theorem 9 is dependent upon Theorem 8; and Theorem 8 is dependent upon Theorem 7. You may carry the proofs back further if you like. Theorem 7 is in turn dependent upon Theorems 1 and 6. And both of these two theorems are directly dependent on the definitions and rules of the system.

The comparison of this hierarchy of theorems with the building of a tower out of blocks is not bad, except that the "tower" of theorems is secure against being toppled over because its base is accepted as a firm foundation. If you will analyze the various proofs from Theorem 1 to Theorem 10 you will see that they can be placed on top of one another just like blocks:



The shaded blocks represent the rules and definitions employed up to this point in building the system. The white blocks are

the theorems built on top of them. Since the shaded blocks cannot be swept out from under, since they represent statements accepted as true at the beginning of construction, the structure erected on them must be entirely secure. Even Theorem 10, high in the air as it is, is firmly fixed, because everything below it is secure either by demonstration or assumption. This diagram illustrates clearly how important Theorem 7 has turned out to be. Of course the tower built in the *Principia* is much more of a skyscraper and its structure is far more complex than this, but the diagram should give an idea of how the building is accomplished.

A New Term Defined and Brought into the System. Conjunction

Every so often a deductive system reaches a point at which the introduction of a new term opens up a whole fresh set of possibilities. The principle is the same as that of the addition of a new word to one's vocabulary, allowing greater refinement in the expression of one's ideas. The word "weld," for example, is relatively new and highly useful.

We have reached such a point in the development of the system of propositions. Suppose we take our primitive ideas again and arrange them in another and somewhat more complex combination:

$$-(-P \vee -Q)$$

This is a combination with properties very useful in dealing with propositions, ones quite familiar but, in their present form at least, difficult to recognize. What will an analysis tell us? This expression should be read: "It is false that either P is false or Q is false or both." Can it be expressed more concisely?

Take it in sections. The part *inside* the parenthesis says that

either P is false or Q is false or both; in other words, one of the two must always be false. But the negative sign *outside* of the parenthesis tells us that this whole idea is false. Hence the whole expression says that *neither* P *nor* Q is false, which is another way of saying that *P is true and Q is true*. When two propositions are both true we may say that they are in a relationship to one another which we shall call "conjunction" and symbolize with a dot (.). Hence we are now in a position to introduce this new term into the system by means of a second definition:

$$P \,.\, Q \,.\!=\!.\, -(-P \vee -Q) \qquad \textit{Definition B}$$

This new idea is obviously convenient, substituting the simple $P \,.\, Q$ in place of the complex $-(-P \vee -Q)$.

You may wonder why the dot was chosen as the symbol for this new relation when we already are using a dot in place of the more cumbersome parenthesis. Is this not going to lead to confusion? In practice, no. The reason is that a conjunction is of a higher order of importance than the other relations in the system and hence acts in itself as a sort of parenthesis. If I write:

$$P \supset Q \,.\, R$$

the dot is more powerful than the horseshoe and hence makes the expression read as though it were written:

$$(P \supset Q) \,.\, R$$

The dot that would ordinarily act as a parenthesis here is combined with the dot of conjunction and written as a single dot.

Let us proceed now to some of the later theorems which the introduction of the relation of conjunction makes possible. They will illustrate clearly the usefulness of this new idea.

Some Old Friends among Propositions

One of the delights of this system of propositions is that all of the things we said in an earlier chapter about arguments derived from a molecular analysis of thinking, things which were at the time quite unrelated to one another and taken altogether too much for granted, can now be stated as theorems and proven on the basis of the fundamental assumptions of the system. We shall find that we were not wrong in what we said at that time, but if any one had asked us why the arguments we accepted as valid *were* valid we could no more have told them than we could show off-hand that the exterior angles of a triangle are equal to the sums of the two opposite interior angles.

In dealing with the implicative argument, for example, we accepted two structures as valid:

$$\begin{array}{c} P \supset Q \\ P \\ \hline \therefore\ Q \end{array} \qquad\qquad \begin{array}{c} P \supset Q \\ -Q \\ \hline \therefore\ -P \end{array}$$

If you will express the structures of these two arguments in the symbolism of the system of propositions, you will find that they can properly be stated as theorems and rigorously proven:

Theorem 11. $p \supset q \,.\, p \,.\supset .\, q$
If p implies q and p is true, then q is true.

Theorem 12. $p \supset q \,.\, -q \,.\supset .\, -p$
If p implies q and q is false, then p is false.

But consider the following two candidates for status as theorems, the implicative argument which denies the antecedent and the one which affirms the consequent:

Theorem (?) X. $p \supset q \,.\, -p \,.\supset .\, -q$
Theorem (?) Y. $p \supset q \,.\, q \,.\supset .\, p$

Without knowing exactly why, we rejected these structures as fallacious. If you are interested in the reason behind this, try to prove them yourself. You will find that it cannot be done. *They do not belong to the system.* They violate the structure of the system the way an Ingersol wheel would violate the structure of a Swiss watch. To attempt to prove them would be like attempting in Euclidean geometry to prove that the sum of the three angles of a triangle is equal to 200°, or in arithmetic, that the square root of 17 is 3.

If you continue your search among the theorems which are susceptible of proof in the new system, you will find other familiar statements:

Theorem 13. $p \vee q \,.\, -p \,.\supset.\, q$

Theorem 14. $p \vee q \,.\, -q \,.\supset.\, p$

These are two forms of the disjunctive argument, which states that if either of two propositions is true and one is known to be false, then the other must be true. And we shall be rewarded for rejecting the other form of the disjunctive argument by noticing that nowhere among the theorems in the system can we find a proof of:

Theorem (?) Z. $p \vee q \,.\, p \,.\supset.\, -q$

This, like X and Y, is an argument which does not belong to the system.

There is an interesting line of thought suggested here which you may want to follow out for yourself. If at the beginning of our system we had defined the horseshoe symbol as meaning *if and only if . . . then,* we should have found ourselves working out a system of which Theorems X and Y would be *bona fide* parts. And if we had taken the sign for disjunction to mean *either . . . or and not both,* Theorem Z would also be part of the system. In that case we would have had to employ a differ-

ent set of rules and our system would exhibit quite a different structure. Work on it a bit, if you like. You will soon discover that a system of propositions employing these more strict definitions will be more limited than the system in the *Principia*. But as a system it will be perfectly good. It will just be a system in which more highly specialized meanings are employed. It would be like a system of geometry in which all triangles had by definition to be isosceles. Such a geometry would be more limited in scope than that of Euclid.

There are two more old friends among the theorems of the system, the ones which cover the two statements of the dilemma, which we studied in Chapter Four:

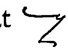
Theorem 15. $p \supset q . r \supset s : p \vee r . : \supset . q \vee s$
Theorem 16. $p \supset q . r \supset s : -q \vee -s . : \supset . -p \vee -r$

By substituting q for s in Theorem 15, and p for r in Theorem 16, you will get the special cases of the dilemma which we found to be psychologically so effective. These two theorems and their special cases are also subject to definite proof and hence are valid within the system we are describing.

Another New Term Defined and Introduced: Equivalence

If any one were mean enough to stop you and ask if you knew the difference between "equals" and "is equivalent to," would you know what to say? We habitually use the two ideas interchangeably. But they do not mean the same thing. Things which are equal are like identical kid gloves, both for the same hand, which can be substituted for each other without altering matters. Things which are equivalent are like a pair of kid gloves, alike in material and size, but one for the left and one

for the right hand. You would notice it if one were substituted for the other.

Now we understand why the equals sign is used in stating definitions. It means that the expressions on either side of the sign are identical in the sense that they say exactly the same thing in different ways, and either side of the equation may be substituted for the other at any time. It tells us that by adding new ideas (and their symbols) to the system we are indulging in a shorthand which makes it easier to write things down. When you are learning shorthand you will find that  equals "logic," and the shorthand is a handier way of writing the word than the longhand way was. All definitions are indications of shorthand devices for expressions which would be more cumbersome if always written in terms of primitive ideas. In geometry "triangle" is shorthand for "a figure bounded by three straight lines in the same plane."

But the idea of *equivalence* is something else. In a country whose police force and courts functioned perfectly, committing a crime would be equivalent to serving a jail sentence. This does not mean that they would be equal. You would not go up to the judge and say: "I have committed a crime, so I do not have to serve a jail sentence." It means that if you commit a crime you will serve a jail sentence, and if you serve a jail sentence you have committed a crime. The relation of equivalence is a reversible implication. Using "$\equiv$" as its symbol, we can define equivalence in this way:

$$P \equiv Q \,.\, = \,.\, P \supset Q \,.\, Q \supset P \qquad \textit{Definition C}$$

Two propositions are equivalent if each implies the truth of the other.

You will notice immediately that this definition is not in terms of primitive ideas. Neither *implication* nor *conjunction* is

primitive. Is there anything wrong? No, because although *implication* and *conjunction* are not themselves undefined, they are in their turn defined by way of ideas that are primitive. Hence, in a final analysis, equivalence also goes back for its meaning to the primitive. If you are interested you can work out a definition of equivalence which will employ none but undefined ideas. But when you get your result you will see the fruitlessness of the effort. The result is so cumbersome as not to show clearly the properties of *equivalence*. The definition quoted above is far more eloquent and just as correct.

This is the last of the defined terms which we shall introduce. It, with *implication* and *conjunction*, opens all of the possibilities we shall care to treat in this exposition.

Some Other Theorems Which You Will Recognize

In drawing designs with pencil and paper we are vaguely aware that the perpendicular is the shortest distance from a point to a given line. But if you are a geometrician, vagueness is vagueness and to be avoided. In our thinking we should want to be equally careful. One of the advantages of the system of propositions is that it brings to exact statement a number of relations among propositions of which we are all vaguely aware, but which are so numerous that they seldom receive concrete expression. The system puts us in a position to state and prove a number of logical principles which we employ in our thinking all of the time.

Suppose we list some of them:

Theorem 17. $p \equiv -(-p)$

The full expression of the *Principle of the Double Negative*. See Theorem 10.

Theorem 18. $-(p . -p)$

This we know as the *Principle of Non-Contradiction.* We know that there is something wrong with our thinking if the same proposition may be both true and false. If in my relations to my family doctor I go on the principle that my actions are completely determined by my physical health, and in my relations to my church I assert that I have freedom of will, then something is wrong with my thinking.

Theorem 19. $p \supset q . q \supset r . \supset . p \supset r$

This is to the molecular analysis what the syllogism *Barbara* is to the atomic analysis. If *I learn to play the cello* implies that *I shall be able to play Mozart quartets,* and *I learn to play Mozart quartets* implies that *I can have a whale of a good time,* then *I learn to play the cello* implies that *I can have a whale of a good time.*

We are constantly relating propositions in this way, indeed this is one of the most familiar of our mental techniques.

Theorem 20. $-(p \vee q) . \supset . -p . -q$

If it is false that either you have aching teeth or you pay enormous dentist bills, then you do not have aching teeth and you do not pay enormous dentist bills. In other words, if a disjunction is false, then neither of its alternatives is true.

Theorem 21. $-(p . q) . \supset . -p \vee -q$

If it is false that you go to the party and insult your hostess, then either you do not go to the party or you do not insult your hostess.

These few samples will give you an idea of the many common relations among the propositions in your thinking. To know them accurately is much better than guessing at them, just as sailing by a chart is better in foggy weather than sailing by guess and by gosh. When we think, we have to do a lot of sailing in fog, human nature being what it is.

Some Theorems You Would Probably Not Recognize

Another of the advantages of working out the system of propositions is that it tells us a host of things which we might otherwise never have known about propositions. There are similar advantages in the field of geometry. In my geometry book I run across this theorem: "In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other upon that side." Now this is not a theorem which I find myself using every day, in fact I doubt if I have ever used it. But if I were a geometrician I think I should have enough curiosity about triangles to want to have it as part of my equipment. Yet without the system I should never have discovered it.

Let us list, then, some of the less obvious theorems in this new system, not because we shall find them particularly useful but because being organisms which are manipulating propositions all of the time we should acquaint ourselves with their possibilities. The concrete applications I shall leave to you:

Theorem 22. $p \supset q \,.\, p \supset r \,.\supset: p \,.\supset.\, q \,.\, r$
Theorem 23. $p \supset q \,.\, r \supset q \,.\supset: p \vee r \,.\supset.\, q$
Theorem 24. $p \,.\, q \,.\supset.\, r :\supset: p \,.\supset.\, q \supset r$
Theorem 25. $p \,.\supset.\, q \supset r :\supset: p \,.\, q \,.\supset.\, r$
Theorem 26. $p \,.\, q \vee r \,.\equiv: p \,.\, q \,.\vee.\, p \,.\, r$
Theorem 27. $p \supset q \,.\equiv: p \,.\equiv.\, p \,.\, q$
Theorem 28. $q \,.\supset: p \,.\equiv.\, p \,.\, q$
Theorem 29. $r \supset -p : p \,.\equiv.\, q \vee r \,.:\supset: p \,.-q \,.\equiv.\, r$

The last one I put in just to be mean, but the others should not surpass your ability by this time to read and understand them.

The System of Propositions and the System of Arithmetic

Probably there is no one who encounters this system of propositions for the first time who is not reminded of the algebra he once studied in school. Both consist of a collection of letters and other symbols. If your experience with algebra was not so painful that even the memory of it makes you shudder, you will find it both interesting and profitable to compare the structure expressed in algebra with the structure of propositions.

You may have done this already. There are several points in the system of propositions at which algebraic equations are forcibly suggested. One has been mentioned already, the case of double negation:

in algebra	$a = -(-a)$
in logic	$p . \equiv . -(-p)$

This suggests the idea of substituting numbers for p and q and r, and arithmetic meanings for ⊃ and . and v. Can this be done successfully throughout the system? In other words, is the system of propositions like the system of arithmetic in structure?

You may be tempted to answer in the affirmative. Other cases will immediately suggest themselves. Rules III and IV look very algebraic:

in algebra	$a + b = b + a$
in system of propositions	$p \vee q . \supset . q \vee p$ (Rule III)
in algebra	$a + (b + c) = b + (a + c)$
in system of propositions	$p . \vee . q \vee r . \supset . q . \vee . p \vee r$ (Rule IV)

As a matter of fact it is suggested in the *Principia* that *conjunction* (.) is logical multiplication and *disjunction* (v) is logical addition. The algebraic equals (=) seems to correspond to the

logical relation of *implication* ($\supset$). Occasionally you will find a theorem in the new system that looks algebraic. For example:

in algebra	$a = a$	
in system of propositions	$p \supset p$	(Theorem 7)

If you look only at these examples it will seem that we are on the point of discovering an important similarity between logic and arithmetic. The likeness is so exact as to be most convincing.

But the fact of the matter is that the two structures are quite different. The one expresses the properties of numbers considered *as quantities*, and the other expresses the properties of propositions considered *as propositions*. "Negation" and "minus" have in common the property of cancellation when doubled, but to say that a proposition is false is entirely different from saying that a number is minus. And likewise with the other supposed similarities. The truth is that the system of propositions is a unique structure, unique to propositions. Rules I and II show clearly why this is the case:

Rule I $P \vee P . \supset . P$

Not by any stretch of the imagination could this be a structural characteristic of numbers. We can never say that $a + a = a$. Numbers do not work that way. Propositions do.

Rule II $P . \supset . P \vee Q$

It would be even more wild to suggest that $a = a + b$.

Our respect for the five rules of the system increases. What they do is to define the fundamental characteristics of propositions so that all of the more complex relations among them can be worked out. There is no better way of getting a thorough appreciation of the structure of propositions than comparing it with the more familiar structure of numbers. Superficially they have similarities, but basically numbers and propositions act quite differently.

If you do not believe that the system of propositions is unique, try to find other meanings for p and q and r and v and $\supset$ and the other symbols. And then see if they hold in the various rules and theorems. You can get a lot of entertainment out of this. Suppose, for example, that you call p and q and r, "red," "yellow" and "blue" respectively. Let the horseshoe mean "is the same color as"; let the disjunction mean that the two colors thus joined are "striped"; and let conjunction mean that the two colors are "mixed." In this case:

Rule I becomes:	Red striped with red is the same color as red.
Rule III becomes:	Red striped with yellow is the same color as yellow striped with red.
Theorem 7 becomes:	Red is the same color as red.

Fine, so far. But you soon get into difficulties. Rule II is a total loss: "Red is the same color as red striped with yellow!" And what would you do with such statements as Rule V, Theorem 1, and Theorem 5? The only reasonable interpretation of – would be "is the complementary color of," but try this in propositions in which – appears and see what happens. Unless you have a better imagination than mine, we shall have to call the structure of the system of propositions unique.

A Martian System of Propositions

From the way in which the definitions and rules of the system are stated it should be clear by this time that in developing the system of propositions we must keep constantly in mind the actual properties and characteristics of propositions *as we use them.*

We have seen that if even slightly different meanings are given to symbols the system must undergo alterations if it is

to be useful. If, for example, we interpret the symbols for disjunction to mean "either . . . or *and not both*," we shall want to drop Rule II. Under the new interpretation this would read: "Rule II: If one proposition is true, then either a second proposition is true or the first is true *and not both*." Even a cursory acquaintance with propositions tells us that this does not hold in the system of our thinking. If it did, we should have to say, for example: "If Paderewski is playing the piano, then either he is whistling or he is playing the piano *and not both*." The point is that we are trying to express the structural relations of *any* two propositions, and there are many which do not perform in this manner. If Paderewski were playing the French horn we should have an entirely different situation. You cannot play the French horn and whistle at the same time. But this is a special case.

It is not imperative to drop the new Rule II. You are free to build any kind of a structure that may please you, play any game that suits your fancy. But one structure will express the character of human thinking in general and one will not. In the field of logic we are interested in one that will, hence we rule out special cases.

One of the logician's most fascinating nightmares consists in imagining a system such as the celebrated Man from Mars might use. Why should we terrestrials have a corner on propositions and their properties? May it not be that Martians use propositions in an entirely different manner, think according to quite other rules? It is difficult for us in our sane moments to follow out this possibility, but let us suppose that Martians use two entirely different realms of thinking, one for Sundays (assuming that there are such things on Mars) and the other for weekdays. This is not so far from what many terrestrials do! And let us suppose that in the Martian system of propositions alternate ones are attached to each realm, just as our numbers are

alternately even and odd. The Martian logician might employ, as primitive, relations such as these between propositions:

$P \boxdot Q$ meaning "P belongs to one system, Q to the other."
$P \odot Q$ meaning "P can tell us about Q."

He could then employ as a definition:

$$P \triangle Q \,.\!=\!.\, -(P \square Q) \qquad \textit{Definition A}$$

Of course the Martian logician might not employ propositions at all in his "thinking." He might use only expletives. I leave you to carry this idea further.

Take even the simple supposition that he employs the same relations and terms that we do. He still has alarming latitude in stating his rules. For example:

Rule I $P \,.\, Q \,.\!\supset\!.\, R$

If two propositions are true, any third is true.

If Greta Garbo is a movie star and Babe Ruth is retired, then tobacco is a filthy weed.

Rule II $P \vee Q \,.\!\supset\!: P \equiv Q \,.\, P \,.\, Q$

If either P is true or Q is true, then P is equivalent to Q and both are true.

If either wishes are horses or beggars may ride, then wishes are horses is equivalent to beggars may ride and wishes *are* horses and beggars *may* ride.

Make sense out of these if you can! Or better still, take an imaginary rocket trip to the moon and work out a system of propositions according to which lunar creatures might "think." Then try to think in that system. When you come back to earth you will be sure that there is no place like home, and your appreciation of our own system of propositions will be at a maximum. Perhaps we should arrange regular trips to Mars or the moon as field work in logic.

Chapter Eleven

THE COPERNICAN SYSTEM OF CLASSES

HAVE you ever wished you might take one of your ancestors for a mile-a-minute spin along a six lane concrete highway? Sixty miles an hour would be enough. If he did not die of fright in the first few minutes he would have a glorious time. And he would find the experience highly instructive. Not that I think the automobile the finest expression of modern civilization. The death rate for which it is responsible is appalling. If, as it happened in the old story books, there were a dragon in the deserts of the southwest who had to be fed on one hundred human beings a day, we should pay a modern St. George plenty to exterminate him. But from the point of view of transportation an automobile is a vast improvement over a horse and buggy. It is far more intricate, but at the same time more delicate and more powerful.

It is hardly likely that Aristotle would die of fright if we could show him what we do to-day with his Ptolemaic analysis of human thinking. But I do think he would have much the same reaction that our ancestor would have if the latter were given the opportunity to examine the mechanism of one of our automobiles. In short, he would be astonished. We have already seen the improvements made by logicians after Aristotle in the study of the syllogism. They correspond, perhaps, to the improvement of the art of the 18th Century carriage maker over the art of making vehicles of transportation in the Athens of

Aristotle's time. We have gone far beyond the carriage maker, and we have also made a great advance in our study of the atoms of thought over the work done on the four figures of the syllogism. We have worked these atoms into a deductive system as powerful in its way as the system of geometry or the system of propositions. Like Aristotle, we have taken the class as our fundamental unit, but we have handled it in a manner which gives more eloquent results. It includes all that Aristotle and the later students of the syllogism have had to say, and a great deal more; just as the modern automobile can do everything that a buggy can do and much besides. It is to Aristotle's system what the Copernican astronomy is to the Ptolemaic.

Though you may not recollect it at the moment, we were given an inkling of what we might do with classes, when we used a new and improved symbolism for them in working out the theory and application of the antilogism. We saw that this new symbolism made clearer various of the properties of A, E, I, and O propositions. The difference in sign, for example, between an E and an I proposition showed immediately why the two were contradictory and the content showed why they alone were subject to simple conversion. Now we are going to study that new symbolism with a vengeance and work out a Copernican system in terms of it.

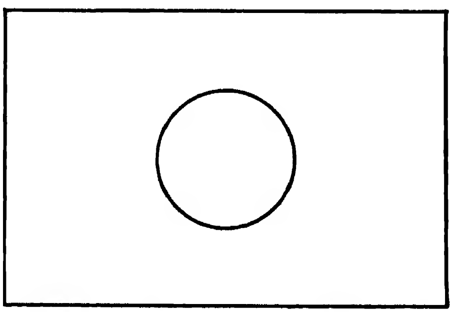
If I Have One Class

Having already worked out the system of propositions, showing carefully the reason for each step, and making clear the way in which a deductive system is developed, we may proceed more rapidly with the establishment of this new system. We are better acquainted with the process.

The idea of a "class" is to this system what the idea of a "proposition" is to the other, an undefined idea. We all know

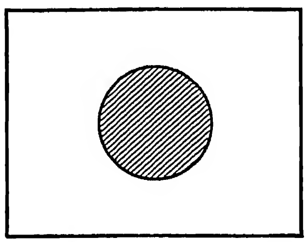
what a class is and we are all familiar with the idea of classification. We have all played the parlor game in which we have to name vegetables, poets, and pieces of furniture beginning with certain letters. There may be classes of things: shoes, ships, and sealing wax are three familiar ones. There may be classes of ideas, as the Seven Deadly Sins or the Seven Liberal Arts. The most familiar classes are classes of things, and these we shall use most often for illustrative purposes. The possibilities of classification are limitless. It would be impossible to gather together any group of objects which could not be classified together. They would belong to the class of "things I have gathered together" if to no other. Shoes and ships and sealing-wax seem fairly independent of one another yet they belong to the class of things the names of which begin with "s." And if you add to them "cabbages and kings" you have the class of things mentioned in a poem by Lewis Carroll. Any group of things can be classified as "physical objects," and so on. There are innumerable types of classification.

As you know already, it is customary to represent the class graphically by drawing a circle. Everything within that circle belongs to that class, and everything outside of the circle is excluded from it. Going a little more carefully into this graphic representation now, we shall place a rectangle around the circle, thus:

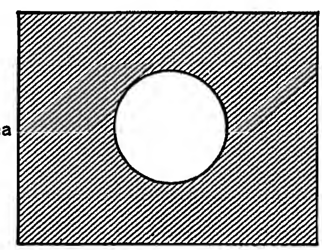


Letting the rectangle enclose everything outside of the class represented by the circle, we now have two ideas to symbolize, (1) the things within the circle, and (2) the things outside of the circle. In this system we choose the first letters in the alphabet to symbolize classes, A, B, C and so on (corresponding to the P, Q, R of the system of propositions). Suppose we call everything within the circle an A. Then everything outside of the circle will be a "not–A." Using the same sign for negation in this system that we used in the system of propositions, we shall write everything outside of the circle as –A. If A is the class of "red things," then –A is the class of "things which are not red." Hence:

A is this shaded area



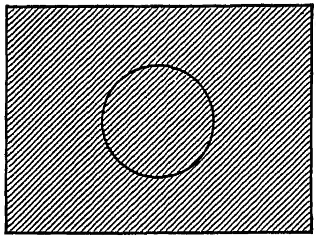
−A is this shaded area



So far so good.

But now suppose we want to speak of *all* of the things inside the rectangle? This would be the class of all classes, the class of everything. The symbol used for everything is 1:

1 is this shaded area

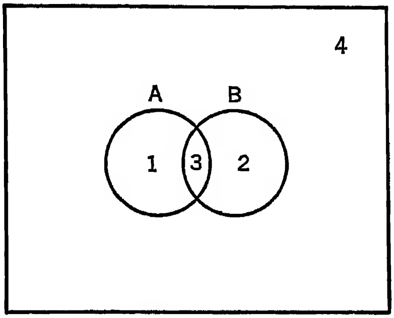


This largest class of all is to the system of classes what infinity is to arithmetic, the largest number. All of the numbers used in counting range between two limits, zero and infinity. And, similarly, 1 represents one of the limits in the realm of classes. What would the other limit be? Well, if some one asked you for the opposite of "everything" you would say, "nothing." The other limit is the class of nothing, the class without members. Is there such a class? If you let 1 stand for objects *and ideas* there could be no null class, for you could mention nothing that would not be at least an idea. You might suppose, for example, that the class of "unicorns" would be a class without members, but surely you have an idea of a unicorn and have seen pictures of them. You can mention unicorns: they *must* be a class!

At this point you are asked to remember something we said when we developed the system of propositions. We are not interested in just a game now, we are interested in a system that will be useful. So we restrict the meaning of 1 (sometimes called "the universe of discourse") to every *thing*, every existent thing. Our system then becomes, as we shall soon see, a useful description of the relations among all classes of existent objects. In this case the class without members has real meaning. Such things as "square-circle," the "Mad Hatter," "United States senators born before 1492," "phoenixes" and so on are null classes: These classes we shall designate by using the appropriate symbol, 0.

If I Have Two Classes

We need some kind of cement in this system, too. We need some ways of joining classes together. Suppose I have two classes; the class A of red things, and the class B of silk things. We know that these two classes overlap thus:

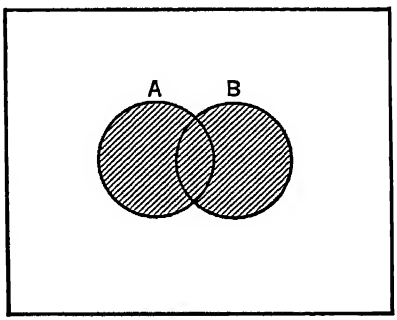


Now some of these areas we know how to symbolize already. For example:

area 1 and 3 can be written as A (red things)
area 2 and 4 " " " " –A (things not red)

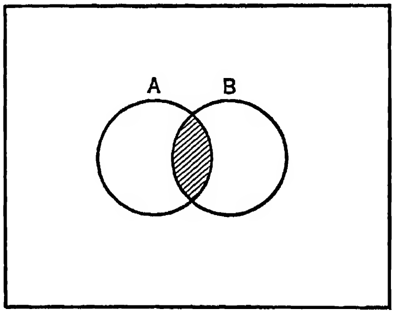
But suppose I want to speak of "things which are either red or silk," area 1 and 2 and 3? How shall I do this? I want to be able to symbolize the class of things that are either A's or B's. The symbol that has been selected for this connection between two classes is +, and "things which are either A's or B's" would be written $A + B$:

A+B is this shaded area



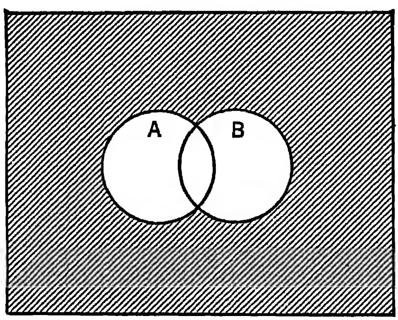
And suppose I want to speak of "things which are both red and silk," the area 3? This can be done by selecting a symbol which shall express things that are both A's and B's. The symbol used for this is $\times$, and "things which are both A's and B's" would be written $A \times B$:

A x B is this shaded area

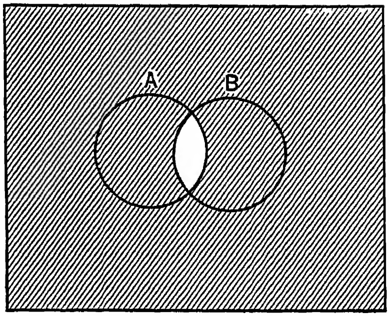


You will readily see that $A + B$ and $A \times B$ are themselves classes, compound classes they are called. Treating them as classes you can use the negative sign in connection with them. If, for instance, you wanted to speak of "things which are neither red nor silk" or of "things which are not both red and silk" you would write:

−(A+B) is this shaded area

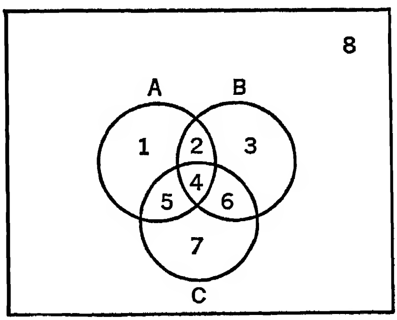


—(A x B) is this shaded area



You will have to forget your algebra and concentrate on these two new meanings for + and ×. You are moving in a different world now. There is nothing I would recommend more urgently than getting to know these two connections among classes so well that when you see the symbols you think of them *before* you think of the algebraic meanings.

There is no better way of improving your acquaintance than by taking a diagram containing three classes, say "red things" and "silk things" and "pajamas":



and finding various ways of expressing symbolically the classes you find in it. How would you symbolize "red silk pajamas," or "things which are red and pajamas but not silk"? And, using A's and B's and C's again, how would you express areas 1 and 3 and 7 and 2 and 5 and 6?

You notice that $A + B$ symbolizes "*either* A *or* B" and that $A \times B$ expresses "*both* A *and* B." *Always use these words in reading the symbols if you want to keep out of trouble.* These words, "either . . . or" and "both . . . and" should remind you of something in the preceding chapter. Do you remember a relation between propositions which said that *either* P *or* Q was true, and another which said that *both* P *and* Q were true? For those who are interested, there is an interesting parallel here:

either P is true or Q is true	P v Q
things that are either A's or B's	A + B
both P and Q are true	P . Q
things that are both A's and B's	$A \times B$

(v) is the disjunctive connection between propositions, and + is called the *disjunctive connection* between classes. (.) is the conjunctive conection between propositions, and $\times$ is called the *conjunctive connection* between classes. The logical properties of the two disjunctions are so much alike that some logicians use the same symbol for both. And the same is true of the two conjunctions. But more about this question later.

Two Relations between Two Classes, One Undefined and One Defined

The word "equals" occurs in every mathematical or logical system with which I am acquainted. It is one of the most fundamental words in our entire language. If we stuck to "red things" and "silk things" we might not have occasion to use it here, but suppose we switch to "equiangular triangles" and "equilateral triangles." If you know anything about geometry you know that these two classes "equal" one another. And by this, of course, we mean that they have the same members. The

two classes are identical. This idea of equality is also undefined within the system of classes, the assumption being that we all know what it means in this connection. At the present time the class of "men who have flown around the world in a day" *is equal to* the class of "men who have reached the summit of Mount Everest and returned to tell the tale"—they are both null classes. The symbol for this is the ordinary equals sign, =.

There is one other relation among classes with which we are all familiar. It will quickly be suggested by the following pairs of classes:

men	and	Englishmen
mammals	and	dogs
paintings	and	frescoes
firearms	and	revolvers

The classes on the right are, we say, "included in" the classes on the left. This idea of class inclusion is one which we are in a position to define in terms of primitive ideas already at our disposal. We shall use *negation* and *conjunction* and *equals* and *null class*. If things which are both dogs and not mammals is a null class, then the class of dogs is included in the class of mammals. If we could find one dog who was not a mammal then the former class would not be included in the latter and, at the same time, the class which contained both dogs and not-mammals would not be null. Hence we may write for any classes, A and B:

$$A < B \,.=.\, A \times -B = 0 \qquad \textit{Defintion A}$$

The symbol used for "included in" always has its smaller end pointed at the smaller class. This makes it easy to remember.

If you are interested in one more parallel between the system of classes and the system of propositions we might suggest that

you compare "included in" with "implies." They look something alike as symbolized. And, as we shall see, they have somewhat similar properties. Both, for example, are transitive relations, each within its own field. But we are getting ahead of ourselves.

The Rules of the Game of Classes

Here we are faced with an embarrassment of riches. The system of classes is older than the system of propositions and men have worked out more variations on its main theme. For our own purposes in outlining the system we shall employ in a slightly modified form the first of three sets of rules developed by Huntington.[1]

You will have the same experience of boredom with these rules as with those in geometry and the system of classes. But you know better now than to be misled by the obvious into believing it unimportant. In logic the more obvious is often the more fundamental.

Rule I $A + 0 = A$

Things that are either pajamas or nothing are pajamas.

Rule II $A \times 1 = A$

Things what are both pajamas and any things are pajamas.

Rule III $A + -A = 1$

Things that are either pajamas or not-pajamas are all things.

Rule IV $A \times -A = 0$

Things that are both pajamas and not-pajamas are nothing.

[1] If you are interested in other sets of rules for the system of classes, you will find Huntington's three sets in Transc Amer. Math Soc., vol. 5 (1904), pp 288-309, and Sheffer's unique set in *Ibid.*, Vol. XIV, pp. 481-488.

Rule V $A + B = B + A$

Things that are either silk or pajamas are the same as things that are either pajamas or silk.

Rule VI $A \times B = B \times A$

Things that are both silk and pajamas are the same as things that are both pajamas and silk.

Rule VII $A + (B \times C) = (A + B) \times (A + C)$

Things that are either pajamas or both silk and red are the same as things that are both either pajamas or silk and either pajamas or red.

This rule is somewhat less obvious than the ones preceding it. If you are in any doubt about it, draw three intersecting circles, label them A and B and C, and compare the areas denoted by the left-hand side of the equation with those denoted on the right. No matter how you draw your circles you will find that this rule always holds.

Rule VIII $A \times (B + C) = (A \times B) + (A \times C)$

Things that are both pajamas and either silk or red are the same as things that are either both pajamas and silk or pajamas and red.

These rules function like others in other systems. They set up a series of connections among the elements of the system which shall be so obvious that no one will wish to question them, yet so significant that the entire system can be developed out of them.

The Nutshell Technique Again

It is just as important here as it was in the system of propositions to get to know the various properties of the game so well that you can recite them while standing on your head or going over Niagara Falls in a barrel. So bare the other cuff or get out another index-card and write down the following·

Counters:	Undefined	A, B, C	classes
		$-$	not
		$+$	either . . . or
		$\times$	both . . . and
		0	null class
		1	class of all classes
	Defined	$<$	included in *Def. A*

Rules:

I $A + 0 = A$

II $A \times 1 = A$

III $A + -A = 1$

IV $A \times -A = 0$

V $A + B = B + A$

VI $A \times B = B \times A$

VII $A + (B \times C) = (A + B) \times (A + C)$

VIII $A \times (B + C) = (A \times B) + (A \times C)$

When you get intimately familiar with these counters and rules you will begin to notice a number of details that had escaped you earlier. For one thing, you will observe that the rules fall neatly into pairs and are more easily remembered as such. When you find yourself reciting these things in your dreams, then you are ready to play the game with assurance and ease.

How Theorems Are Proved in the System of Classes

As in all deductive systems, the first theorems are quite simple. But perhaps this is just as well. It gives us an opportunity to concentrate on the technique of proof. We shall take as our first theorem one that might well have been a rule of the game.[2] Indeed it reminds us forcibly of the very first rule in the system of propositions (can you recite it without looking back?). No one will doubt that things that are either pajamas or pajamas are pajamas. But the general underlying idea is not expressed among the rules, so we shall have to prove it:

[2] The order and numbering of these theorems are those of the present author.

Theorem 1

To prove: $a + a = a$

Looking back to Rule I we see that $a + o = a$. But Rule IV tells us that $a \times -a = o$. Hence we may substitute $a \times -a$ in the place of the zero in Rule I. This would give us:

$$a + (a \times -a) = a \qquad \text{(i)}$$

Now Rule VII tells us that for *any three classes* $A + (B \times C)$ is the same as $(A + B) \times (A + C)$. You will notice that in (i) we have three classes related to one another in exactly the manner in which the three classes are related in the first half of Rule VII. To wit:

$$\begin{array}{ccccc} A & + & (B & \times & C) \\ \downarrow & & \downarrow & & \downarrow \\ a & + & (a & \times & -a) \end{array}$$

Note that the plusses and crosses and parentheses are in exactly the same places. Therefore

since $A + (B \times C)$ is the same as $(A + B) \times (A + C)$
$a + (a \times -a)$ is the same as $(a + a) \times (a + -a)$
Rule VII

and we may substitute the latter for the former in (i), getting:

$$(a + a) \times (a + -a) = a \qquad \text{(ii)}$$

But reference to Rule III shows that $a + -a = 1$, so we may substitute 1 for $a + -a$ in (ii), and have:

$$(a + a) \times 1 = a \qquad \text{(iii)}$$

And Rule II tells us that any class conjoined with 1 is the same as itself. Now $a + a$ is a class. Hence $a + a$ conjoined with 1, as in (iii), will be the same as $a + a$; and $a + a$ may be substituted for $(a + a) \times 1$ in (iii), giving

$$a + a = a$$

Q.E.D.

Here, as in geometry or in the system of propositions, we must get accustomed to the shorthand of proof. Having followed each step in this demonstration carefully, we shall now write it more concisely:

To prove: $a + a = a$

$a + 0 = a$	by I
$a + (a \times -a) = a$	by IV
$(a + a) \times (a + -a) = a$	by VII
$(a + a) \times 1 = a$	by III
$a + a = a$	by II

Q.E.D.

Just to accustom you to the nature of proof in this system, here are two more simple demonstrations. If followed carefully, they will help make you feel at home in the system:

Theorem 2

To prove: $a \times a = a$

$a \times 1 = a$	by II
$a \times (a + -a) = a$	by III
$(a \times a) + (a \times -a) = a$	by VIII (do you follow this one?)
$(a \times a) + 0 = a$	by IV
$a \times a = a$	by I

Q.E.D.

Theorem 3

To prove: $a + 1 = 1$

$a \times 1 = a \quad (1)$	by II
$(a + 1) \times 1 = a + 1$	subst. $a + 1$ for a in (i)
$1 \times (a + 1) = a + 1$	by VI
$(a + -a) \times (a + 1) = a + 1$	by III
$(a + -a) \times 1 = a + 1$	by VII (get it?)
$a + -a = a + 1$	by II
$1 = a + 1$	by III

Q.E.D.

Similar to these is the proof of Theorem 4, which states that $a \times o = o$. Can you find it? I advise you to start by asserting $(a \times o) + o = a \times o$, a special case of Rule I. When you understand thoroughly the reason for each of the steps in these demonstrations you will have made real progress. One of the best ways of finding out whether or not you really do understand what goes on is to try to explain it to some one else. It is one of those things your best friend *will* tell you!

A More Difficult Proof

Any one who has studied algebra will find these proofs easier to understand than those in the system of propositions, for in many ways they are more like arithmetic manipulations than those that deal in p and q and r. But there are certain other respects in which all three systems are very much alike.

For example, we found in the system of propositions a *Principle of the Double Negative* that was very much like the principle in algebra according to which two minus signs became a plus sign. We found that any proposition, p, was equivalent to $-(-p)$. This same general principle applies in the system of classes. In this case the negative is the negative of a class. And two such negatives cancel one another: $-(-a) = a$. We are so accustomed to the idea that we think it quite simple. But, as is often the case, this simple idea requires a rather complex proof. The proof of the principle of the double negative in the realm of classes employs a technique not found in the three proofs already quoted. Hence it is one we should study in detail. It starts out as two separate proofs which come together only at the end:

Theorem 5

To prove: $-(-a) = a$

$a \times -a = 0$	(i)	by IV
$-a \times -(-a) = 0$	(ii)	subst. $-a$ for a in (i)
$a + (-a \times -(-a)) = a$		subst. the left-hand side of (ii) for 0 in I
$(a + -a) \times (a + -(-a)) = a$		by VII
$1 \times (a + -(-a)) = a$		by III
$a + -(-a) = a$	(iii)	by II and VI

Now the second part of the proof. Starting again:

$a + 0 = a$	(iv)	by I
$-(-a) + 0 = -(-a)$		subst. $-(-a)$ for a in (iv)
$-(-a) + (a \times -a) = -(-a)$		by IV
$(-(-a) + a) \times (-(-a) + -a) = -(-a)$		by VII
$(-(-a) + a) \times 1 = -(-a)$		by III and V
$-(-a) + a = -(-a)$	(v)	by II

So far we have two proofs:

the 1st tells us that $a + -(-a) = a$ (iii)

the 2nd tells us that $-(-a) + a = -(-a)$ (v)

But from Rule V we know that:

$$a + -(-a) = -(-a) + a$$

Therefore $-(-a) = a$

Q.E.D.

No need to inform you that a logician's intuition is just as necessary for working out the demonstrations here as in the system of propositions! When you see it all written down it looks fairly direct, but the men who first worked on these proofs must have had as many moments of despair as Columbus did on the way to the West Indies.

Still Another Type of Proof

As if to say that things are not difficult enough to understand as they are, this system boasts another type of theorem which requires its own kind of proof. The five theorems we have quoted have all been statements of equalities. All we had to do was to state the equality in the symbolism of the system and then prove it. But now we encounter other theorems in which it is stated that if two classes are related in a certain manner, then they are also related in another manner. How would we go about proving such a statement?

For example, there is a theorem to the effect that *if* things which are both a's and not b's is a null class, *then* the things which are both a's and b's is equal to the class of things which are just a's. Notice the "if . . . then . . ." part of this statement. A condition is set up and, given this condition, one may show that another relationship holds. Perhaps a concrete case will help here. *If* things which are both elephants and do not have trunks is a class without members, in other words *if* there are no things which both are elephants and do not have trunks, *then* things which both are elephants and have trunks are the same as things which are elephants. This sounds reasonable. And so it is. Here is the proof.

Theorem 6

To prove: If $a \times -b = 0$, then $a \times b = a$

$a \times 1 = a$ (i)	by II
$a \times (b + -b) = a$	subst. b for A in III and applying III to (i)
$(a \times b) + (a \times -b) = a$ (ii)	by VIII

But the condition we wish to fulfil in this case is that $a \times -b = o$ and we can fulfil it by substituting o for $a \times -b$ in (ii). Hence

$(a \times b) + 0 = a$	
$a \times b = a$	by I

Q.E.D.

This is like other proofs except that at an appropriate stage in the proceedings the condition being fulfilled is introduced as one of the assumptions and then the demonstration is carried to the desired conclusion.

These five proofs should give an adequate idea of how the system of classes is developed. There is nothing here that differs in principle from the technique of either geometry or of the system of propositions. As we have already said, it is just like building a tower out of blocks. Each block as it is added to the structure is dependent for its support upon those on which it rests, and these in turn are supported by blocks lower down, and so on until we come to the blocks at the bottom of the pile, which are secure because they are so obvious that we are willing to accept them as the assumptions of the system. Given one kind of foundation you can build one style of tower; given another, you can build another. What you can do as an architect of propositions or of classes depends entirely on the characteristics of the blocks at the bottom.

A Unique Characteristic of the System of Classes

Each deductive system seems to have a personality all its own. It is characteristic of the number system, for example, to be made up of even and odd terms. In the system of propositions you might select the principle of non-contradiction as the outstanding property. We shall see that in many respects the system of classes is closer to the number system than is the system of propositions. Just the similarity in the symbols employed would suggest this, but there are also other and more fundamental likenesses.

Numbers can be divided into pairs; even and odd, even and odd, even and odd, and so on. It just happens that the theorems

in the system of classes can also be divided into pairs. Neglecting proofs now, for the sake of mobility, let us give statements of a number of theorems and see in what sense they can be grouped in two's. Looking backward for a moment before we go ahead, however, we notice that Theorems 1 and 2 are much alike:

$$a + a = a$$
$$a \times a = a$$

The only noticeable difference is that between the $+$ and the $\times$. Are there other propositions which differ only in respect to these two signs? Yes, there are. If you look in any book in which the system is described fully you will find the following:

Theorem 7: $a + (b + c) = (a + b) + c$
Theorem 8: $a \times (b \times c) = (a \times b) \times c$

Theorem 9: If $a = b$, then $a + c = b + c$
Theorem 10: If $a = b$, then $a \times c = b \times c$

Theorem 11: $a + (a \times b) = a$
Theorem 12: $a \times (a + b) = a$

Theorem 13: $a + b = -(-a \times -b)$
Theorem 14: $a \times b = -(-a + -b)$

And if you continue to look for pairs you will find something even more interesting. Consider the following theorems and rules in their groupings:

Theorem 3: $a + 1 = 1$
Theorem 4: $a \times 0 = 0$

Theorem 15: $0 = -1$
Theorem 16: $1 = -0$

Rule I $A + 0 = A$
Rule II: $A \times 1 = A$

Rule III: $A + -A = 1$
Rule IV: $A \times -A = 0$

The difference between these pairs in that the $+$ changes to $\times$ *and* the o changes to 1, and *vice versa*. Not only this, but there are wheels within wheels. In the theorems and rules quoted you will find that *if you change all minus signs to positive signs and all positive signs to minus signs:*

Theorem 13 becomes Theorem 14
Theorem 14 becomes Theorem 13

Theorem 15 becomes Theorem 16
Theorem 16 becomes Theorem 15

And if you change all signs, interchange $\times$'s and $+$'s and o's and 1's:

Rule III becomes Rule IV
Rule IV becomes Rule III

In general we may say that if in *any* theorem of the system of classes the $+$'s and $\times$'s are interchanged, and the o's and 1's (if there are any) change to 1's and o's respectively, you will get another theorem equally valid. Thus for any valid theorem in the system there is a companion to it, just as in arithmetic for every even number there is an odd number greater by one which is its companion. In short, each theorem in the system of classes is one of a pair of twins, not identical twins to be sure, but nevertheless twins.

Some Propositions in the System which Should Be Familiar

There are other interesting things about pairs of statements in the system of classes. If I should present you with the following pairs your reaction would be what it should be:

$$a \times -b = 0 \qquad\qquad a \times b = 0$$
$$a \times -b \neq 0 \qquad\qquad a \times b \neq 0$$

In each of these pairs one of the statements will always be true and the other always false no matter what *a* and *b* stand for. In the first pair, for example, the conjunction of *a* and *–b* is a class which either has members or does not have members. There is no other possibility. And the same with the conjunction of *a* and *b*. If your memory is good you will see what I am driving at. These four statements are the familiar A, E, I, and O propositions in disguise:

A—All a is included in b $\quad a \times -b = 0$
If everything that is an *a* is included in things which are *b's*, then things which are both *a's* and *not–b's* will be zero.
E—All a is excluded from b $\quad a \times b = 0$
I—Some a is included in b $\quad a \times b \neq 0$
O—Some a is excluded from b $\quad a \times -b \neq 0$

And we see immediately, in this new symbolism, why A and O propositions are contradictory, and why E and I propositions are also contradictory. If one expression is equal to zero then it cannot not be equal to zero. If one contradictory is true the other cannot be.

There are many things we can see more clearly now than before. Another is the process of conversion. Rule VI tells us that $A \times B = B \times A$, for any class. On the basis of this rule we can prove the following theorem:

Theorem 17: If $a \times b \neq 0$, then $b \times a \neq 0$
This is a statement to the effect that if some *a's* are *b's*, then some *b's* are *a's*. In other words, an I proposition may be converted *simpliciter*.

And two other theorems can be proven, two which will express the conversion of E and A propositions:

Theorem 18: If $a \times b = 0$, then $b \times a = 0$ (E to E)
Theorem 19: If $a \times -b = 0$ and $a \neq 0$, then $b \times a \neq 0$ (A to I)

But none of the following theorems, all of which attempt a conversion of an O proposition, can be found in the system:

Theorem (?) *X:*	If $a \times -b \neq O$,	then $b \times -a = o$	(O to A)
Theorem (?) *Y:*	If "	then $b \times a = o$	(O to E)
Theorem (?) *Z:*	If "	then $b \times a \neq o$	(O to I)

An O proposition cannot be converted.

Not only this, but every one of the properties of the Square of Opposition can be stated as a theorem in the system of classes, and every one of these theorems can be demonstrated to be true without question.

Again we come to the realization that we took much for granted in the early chapters of the book, and are only now beginning to understand the more fundamental reasons behind them. We never doubted the properties of the Square or the processes of conversion, but now we can do better, we can prove them. This process of finding proofs among the theorems of the system of classes is something like taking apart a watch. If you are given a watch you do not doubt that the long hand will tell of minutes and the short one indicate the hours. That is one of the properties of a watch. But *why* should they do this? We do not know the answer until we take the thing apart and see how the various wheels are geared one to another. In seeing the Square of Opposition as part of the system of classes we are really finding ourselves able to take the Square apart and see how it works. The system provides us with a more fundamental analysis than we could otherwise have employed. The difference, of course, is that none of us got the watch together again, whereas it is comparatively easy to reconstruct the Square out of the appropriate theorems.

Aristotle and the System of Classes

If Aristotle was as broadminded as we think he was, it would be a treat to watch his ghostly face if he could study this system we have been developing. He would at first be amazed. He would discover that the theory of classes on which he had written so many words and to deal with which he had worked out so many technical devices, could be greatly simplified. We remember that he established his four syllogisms of the first figure by a *dictum de omni et nullo* and that the other syllogistic arguments could be "reduced" to the first figure in one of three ways, simple conversion, conversion *per accidens*, or by contradiction. We remember that the pattern of the syllogism was interesting but (to put it mildly) intricate.

Aristotle would be quick to see that everything he took so much trouble to say can be found in just *three* of the theorems of the new system! These three theorems are the following:

Theorem 20: If $a \times -b = o$ and $b \times -c = o$, then $a \times -c = o$

Theorem 21: If $a \times b = o$ and $c \times b \neq o$, then $c \times -a \neq o$

Theorem 22: If $a \times -b = o$ and $a \times -c = o$ and $a \neq o$, then $c \times b \neq o$

The proofs of these three theorems are not easy. You may try them if you like, and I will give you the hint of saying that you will best proceed by employing one of those proofs that go off in two directions and then meet dramatically at the end. You will probably prefer to take my word for the validity of these statements. Anyway, I have boasted about these theorems to Aristotle's ghost and I am anxious to make good my words.

You, in company with Aristotle's ghost, probably do not see how all nineteen of the syllogisms can be included under those

three theorems. Well, you are in good company. But Theorem 20 is nothing less than your old friend Barbara in disguise. You do not believe it? Follow me.

Barbara		*Barbara in disguise*
All M is P	which we now know	$b \times -c = o$
All S is M	can be rewritten	$a \times -b = o$
All S is P	as:	$a \times -c = o$

Just take these symbols on the right-hand side and put them single-file with an "If . . . then . . ." relating them instead of a line, and you will have an exact statement of Theorem 20.

Not only this, but Theorem 20 covers Celarent, Cesare, Camestres and Camenes, too; all of the syllogisms in which a universal conclusion is drawn from two universal premises. You are now more sceptical than ever. All right. If you do not believe me, select any one of these and I will show you. Suppose you take Camestres, in the second figure. How would Camestres be written in the system of classes?

Camestres	*Camestres in disguise*
All P is M	$c \times -b = o$
No S is M	$a \times b = o$
No S is P	$a \times c = o$

When written in a straight line the disguise becomes:

$$\text{If } a \times b = o \text{ and } c \times -b = o\text{, then } a \times c = o \qquad \text{(i)}$$

Now I will grant that this does *not* look like Theorem 20. But perhaps if we play according to the rules of the system we can do something about it. Rule VI tells us that for any propositions A and B, $A \times B = B \times A$. Hence changing the middle expression according to this Rule, (i) becomes:

$$\text{If } a \times b = o \text{ and } -b \times c = o\text{, then } a \times c = o \qquad \text{(ii)}$$

This looks more like Theorem 20 than did (i), but still not enough. We know enough about the system to realize that b and c stand for any classes. So let us substitute $-b$ for b and $-c$ for c in (ii). When we do this we shall get:

$$\text{If } a \times -b = o \text{ and } -(-b) \times -c = o, \text{ then } a \times -c = o \qquad \text{(iii)}$$

But the double negative is an old friend of ours by this time. We know that two minus signs cancel one another. Hence (iii) can be written:

$$\text{If } a \times -b = o \text{ and } b \times -c = o, \text{ then } a \times -c = o$$

By a series of valid steps we have translated (i) until it is exactly in the form of Theorem 20. Camestres is another form of Barbara. If you care to take the trouble you can make similar translations of Celarent, Cesare and Camenes to Theorem 20.

Of course you have now guessed that Theorem 21 covers all of the syllogisms in which a particular conclusion (notice the $\neq$ *after* the "then") is drawn from a universal and a particular premise (note the $=$ and the $\neq$ *before* the "then"). Darii, Ferio, Festino, Baroco, Disamis, Datisi, Bocardo, Ferison, Dimaris, and Fresison: all of these arguments are packed into the nutshell of Theorem 21. It takes some manipulation to show that this is so in all ten cases. But I think that by now you may be willing to accept my word in the matter.

Theorem 22 is different from 20 and 21. You will notice that it adds a third premise to the first two, a premise which states that the class common to the first two premises, class a, is not a null class. Three premises! How can this be a form of syllogistic argument when we defined the syllogism as having but *two* premises? But perhaps you will remember the four black sheep of Aristotle's herd of syllogisms; Darapti, Felapton, Fesapo and Bramantip. You may recollect that these are the ones in which the word "some" appears in the conclusion *but*

only in the conclusion. And do you remember that we discovered that the word "some" implies that the classes in question actually exist, and that if the existence of these classes is not assumed in the premises the syllogisms are not valid? Well, that third premise in Theorem 22 asserts that the classes spoken of in the premises *do* exist, and under this condition the four black sheep become valid. Theorem 22, then, is the statement of these last four of Aristotle's nineteen syllogisms, in their valid form. Neat, isn't it!

Just one more point. Even the latest aspect of the theory of the syllogism, one which Aristotle did not even suspect, is given expression in the system of classes. Suppose we take Theorems 20 and 21 and contradict the conclusions of each:

Theorem 20 becomes If $a = -b = 0$ and $b \times -c = 0$, then $a \times -c \neq 0$

Theorem 21 becomes If $a \times b = 0$ and $c \times b \neq 0$, then $c \times -a = 0$

In both cases three conditions are fulfilled:

1. there are two equalities and an inequality
2. the two classes in the inequality appear once each in the equalities with the same sign as in the inequality.
3. the two equalities contain the third class, once positive and once negative.

This should strike a responsive note. These are the rules of the antilogism test! Theorems 20 and 21 cover the fifteen pure white sheep of the syllogism herd, and if their conclusions are contradicted they meet exactly the same requirements. These are the requirements of the fifteen syllogisms which are valid under all conditions. There is nothing in the theory of the syllogism that escapes us when it is translated into the system of classes and, once you get accustomed to the system, believe it or not, the structure of these arguments is much clearer.

Some Things That Aristotle Did Not Know about Classes

Aristotle would have been amazed at the system of classes for more reasons than one. Not only does the system put concisely all of the characteristics of the structure of syllogistic argument, but it also leads us to a whole host of relations among classes only a few of which we should otherwise have known, and then only by accident. It is just like the systems of geometry and of propositions in this respect. It allows us to assert and prove a large number of statements which tell us a wealth of things about classes and of the ways in which they fit together. We shall list a number of such theorems and leave the reader to work out for himself their significances:

Theorem 23: $0 < 1$
Theorem 24: $0 < a$
Theorem 25: $a < 1$

Theorem 26: $a = b$ is equivalent to $-a = -b$
Theorem 27: $a + b = 0$ is equivalent to $a = 0$ and $b = 0$
Theorem 28: $a \times b = 1$ is equivalent to $a = 1$ and $b = 1$
Theorem 29: $a \times -b = 0$ is equivalent to $b + -a = 1$
Theorem 30: $a \times -b = 0$ is equivalent to $a \times b = a$
Theorem 31: $a \times -b = 0$ is equivalent to $a + b = b$
Theorem 32: $a < b$ is equivalent to $-b < -a$

Theorem 33: $a + b = a + (b \times -a)$
Theorem 34: $a \times b < a + b$
Theorem 35: $a \times b < a$ and $a \times b < b$

Theorem 36: If $a < c$ and $b < c$, then $a \times b < c$
Theorem 37: If $a < c$ and $b < c$, then $a + b < c$
Theorem 38: If $a < b$ and $a < c$, then $a < b + c$

Theorem 39:[3] $-(-b + c) + (-(a + b) + (a + c)) = 1$

[3] Taken from Chapman and Henle *The Fundamentals of Logic* (Scribners), p. 214.

Theorem 32 is one that will get you out of many difficulties in handling classes if you will remember it. Others are fairly complex, but important additions to our equipment in dealing with thinking according to the atomic analysis.

Comparisons may be odious, but I am willing to suggest that handling human thinking according to this new symbolism is as much more convenient than handling it according to Aristotle's subject-predicate symbolism as our arabic numerals are more convenient than the old Roman ones were. You can imagine, perhaps, going into a store and calculating the cost of your purchases, and your change, in Roman numerals. You might even try it some time if you can find a coöperative salesgirl. But think of the mess a modern bank would be in if it were restricted to I's and V's and X's and L's and D's and M's. Mind you, there would be nothing wrong about using Roman numerals, but there are many calculations which it would be impossible to make with them. The same is true in logic. There is nothing wrong with the subject-predicate symbolism when it is thoroughly understood. It just is neither as delicate nor as powerful an instrument as the symbolism of the system of classes. If you will compare carefully what we have been able to do in this chapter with what was done in Chapters Two and Three you will very quickly appreciate the truth of this.

An Interesting Chart

Having described the system of classes in some detail, let us now step back a few paces and look at it in perspective. You have often noticed that in taking a picture of an unfamiliar object the photographer knows enough to place beside it some other object whose size is known. A picture of a new flower will have an inch rule beside it. A photograph of a recently dug-up stone carving will show a man leaning awkwardly against it. When

we stand off from the system of classes in order to see it as a complete system, we discover that it is surrounded by other deductive systems of various types. Some are at quite a distance; that is, there are not many points of similarity except the fact that they *are* deductive systems; checkers, bridge, and other games. Systems like that of geometry are closer because they are more similar. If we look carefully we shall see that there are two systems which stand on either side of the system of classes, more companionable than the others because very like it in their general characteristics, algebra and the system of propositions. This proximity is as helpful to the logician as the inch rule is to the botanist or the figure of the man to the archeologist.

This helpfulness can be well illustrated by means of a chart in which the elements of the systems of algebra and of classes and of propositions are shown together. The likenesses are quite striking:

Algebra	*System of Classes*	*System of Propositions*
a, b, c, x, y, z,	a, b, c,	p, q, r
$+$	$+$	v
$-$	$-$	$-$
$\times$	$\times$	$\cdot$
$=$	$=$	$\equiv$ and sometimes $\supset$
$<$ $>$	$<$	$\supset$
0	0	is false
∞	1	is true

Algebra deals exclusively with quantities. Judging by the elements which enter into the system it seems that the properties

of classes will be more like the properties of numbers than are those of propositions. Yet there are certain relationships, as for example *negation* and *disjunction*, which seem to run through all three systems. There are differences, of course, for if there were not the systems would be the same. A study of this chart with the idea of trying to see the likenesses and differences in the meanings of the various symbols will take you a long way toward understanding what the logician is trying to do when he fools around deductively with classes and propositions. Examine it carefully.

Algebra and the System of Classes

But such a chart is only a beginning. The character of a system is not actually determined until the rules of the game have been set down. The chart shows only that the properties used in playing the game are much alike. Let us see what we can discover from a comparative study of the rules of these three systems.

On the right-hand of classes stand numbers. Four of the eight rules in the system of classes remind one forcibly of equations more familiar in the field of algebra. In fact, if given algebraic meanings they *are* true equations of numbers.

System of Classes	*Algebra*
$A + 0 = A$	$a + 0 = a$
$A + B = B + A$	$a + b = b + a$
$A \times B = B \times A$	$a \times b = b \times a$
$A \times (B + C) = (A \times B) + (A \times C)$	$a (b + c) = ab + ac$

And, as we should now suspect, there are many theorems in the system of classes that can be given true algebraic interpretations. Note the following samples:

	System of Classes	*Algebra*
Theorem 3:	$a + 1 = 1$	$a + \infty = \infty$
Theorem 5:	$-(-a) = a$	$-(-a) = a$
Theorem 7:	$a + (b + c) = (a + b) + c$	$a + (b + c) = (a + b) + c$
Theorem 9:	If $a = b$, then $a + c = b + c$	If $a = b$, then $a + c = b + c$
Theorem 23:	$0 < 1$	$0 < \infty$
Theorem 38:	If $a < b$ and $a < c$, then $a < b + c$	If $a < b$ and $a < c$, then $a < b + c$

But there are many more theorems in the system which have no connection with algebra. We encounter them at the very beginning of the list. For example, Theorems 1 and 10:

$a + a = a$
with classes but *not* with numbers: $(3 + 3 = 6)$

$a + (a \times b) = a$
with classes but *not* with numbers: $(3 + (3 + 4) = 10)$

In fact, when it comes to the theorems of the system it is more the exception than the rule that they can be given an algebraic interpretation which will be correct for numbers.

The difficulty, of course, is caused by the four rules of the system of classes which are *not* algebraic in their possibilities. In fact the chief difference between classes and numbers emerges from a study of these four rules:

$A \times 1 = A$
This *looks* algebraic but is not. In algebra it becomes $A \times \infty = A$, palpably false.

$A + -A = 1$
Classes are such that if any one is connected disjunctively with its negative, the resulting class includes all classes.

But it is not true of numbers that any one added to its negative equals infinity. It equals just the opposite, zero.

$A \times -A = 0$ For all classes, a class conjoined with its negative is a class without members.

But in algebra any number multiplied by its negative gives minus its square. *The square of a number is a phenomenon in algebra that has no counterpart in the system of classes.*

$A + (B \times C) = (A + B) \times (A + C)$

This, too, is a property of classes which numbers cannot boast. $3 + (4 \times 5)$ equals 23. But $(3 + 4) \times (3 + 5)$ equals 56.

It is not difficult to see, then, that the rules and theorems of the system of classes give us the unique properties of classes. They tell us how classes act in one another's company. Numbers act in a way which is superficially similar but fundamentally quite different, and hence give quite a different structure when the system which they compose is worked out.

The System of Propositions and the System of Classes

On the left hand of classes stand propositions. Between these two systems there are also interesting comparisons. Among the rules and theorems for the two are some that are very much alike, for example:

System of Classes

Rule V: $A + B = B + A$
Theorem 5: $-(-a) = a$

System of Propositions

Rule III: $P \vee Q \,.\supset.\, Q \vee P$
Theorem 10: $p \,.\supset.\, -(-p)$

If we wish to grub around we shall find many statements in either system which are reminders of similar statements in the

other. Without attempting to make references by number because many of these statements now appear for the first time, we can produce quite a list of similarities:

System of Classes

$$a + (b \times c) = (a + b) \times (a + c)$$
$$a + a = a$$
$$a \times b = -(-a + -b)$$
$$a \times (b \times c) = (a \times b) \times c$$
$$\text{If } a = b, \text{ then } a + c = b + c$$
$$\text{If } a < b \text{ and } b < c, \text{ then } a < c$$

System of Propositions

$$p \vee (q \,.\, r) \,.\supset.\, (p \vee q) \,.\, (p \vee r)$$
$$p \vee p \,.\supset.\, p$$
$$p \,.\, q \,.\supset.\, -(-p \vee -q)$$
$$p \,.\, (q \,.\, r) \,.\supset.\, (p \,.\, q) \,.\, r$$
$$\text{If } p \supset q\text{, then } p \vee r \,.\supset.\, q \vee r$$
$$\text{If } p \supset q \text{ and } q \supset r, \text{ then } p \supset r$$

This is really just a beginning. We could go on almost indefinitely. There is a whole range of analogies that can be drawn between the two systems.

The problem of finding in the system of propositions meanings analogous to those intended by the class symbols 0 and 1 is more difficult. Someone who was very ingenious discovered that if the 0 is interpreted as meaning "is false" and 1 is taken to mean "is true," the parallels between the two systems are greatly extended. Using these interpretations:

$$a \times -a = 0 \text{ becomes } (p \,.\, -p) \textit{ is false}$$
$$a + -a = 1 \text{ becomes } p \vee -p \textit{ is true}$$

In short, Rule III and Rule IV of the system of classes become the Principle of Excluded Middle and the Principle of Non-

Contradiction, respectively, of the system of propositions! Any one who wishes to follow out this line of comparison will be rewarded by a number of interesting discoveries for which, unfortunately, we have not time here. But we cannot refrain from challenging the reader to interpret Theorem 39 of the class system in terms of propositions and their relations:

$$-(-b + c) + (-(a + b) + (a + c) = 1 \text{ becomes}$$
$$-(-q \vee r) \vee (-(p \vee q) \vee (p \vee r) \textit{ is true}$$

Is it? [4]

It may safely be said that if you will make your comparisons correctly there is nothing in the system of classes that cannot be given an analogous meaning in the system of propositions. This should not be altogether a surprise, for both systems are analyses of the same process—the process of thinking. One analyzes it atomically and the other analyzes it molecularly. In handling certain problems the one will be a more useful system than the other, as we should expect. The situation is like that in dealing with equations of two dimensional figures. If you employ the symbolism of the familiar Cartesian coördinates (x and y) you will get as the equation of the circle, $x^2 + y^2 = r^2$. But if you employ the less familiar polar coördinates (the length of a line and its angle of rotation) the equation for the circle will be $r = k$. Here polar coördinates have the advantage of simplicity. But when it comes to finding the equation for the ellipse the advantage is in the other direction.

It would be a mistake to suppose, however, that because statements in the two systems are analogous they say the same thing. This is not at all the case. The one system deals with *classes* and the other deals with *propositions;* the one deals with the *properties of classes* and the other deals with

[4] The suggestion of this comparison is taken from Chapman and Henle's *The Fundamentals of Logic* (Scribners), p. 214.

the *properties of propositions*. For example $-(-a) = a$ tells us that the negative of the negative of a class equals that class. Now it happens that $-(-p) . \supset . p$, the negative of the negative of a proposition implies the proposition. But classes are classes and propositions are propositions, and the negatives of the two are quite different things. There is a similarity of structure here but not of meaning. Again, $a + -a = O$ and $-(p . -p)$. Structurally there is a similarity, but they tell entirely different stories.

A Helpful Interpretation of the Symbols of the System of Classes

Can we take the a's and b's and c's of the system of classes and give them meanings entirely outside of the field of logic, then find appropriate interpretations for the relations and manipulations? I think you will have great difficulty in doing so. A system of colors will not work here any more than it will work with the system of propositions. The long and short of the matter is, of course, that this system develops the properties *of classes* and classes are practically unique.

I say "practically" unique because, fortunately for every one who ever studied logic, there *is* one other set of interpretations which can be given the symbols of this system which *will* work. Did it ever occur to you how difficult it would be to study logic if we could not draw diagrams? When we have been talking about syllogisms and classes we have often taken refuge in drawing circles as a way of illustrating the things we have been trying to explain. It is lucky that we *may* draw circles. If you will remember how much you have depended on them for help I am sure that you will agree with me. Now it just happens that if we let *a* and *b* and *c* stand for areas of circles and then let:

$a + b$	mean "the area included in both circles, a and b"
$a \times b$	mean "the area common to the circles a and b"
$-a$	mean "the area outside of the circle a"
$a = b$	mean "circles a and b enclose the same area"

we have a system of areas which corresponds exactly in its properties with the system of classes! If you do not see what a stroke of luck this is, try sometime to explain the properties of classes to a friend *without* drawing circles.

You have observed, too, that we did not employ illustrative diagrams in talking about the properties of propositions. Why is this? I leave the problem to you, cautioning you only that the most obvious answer is *not* the correct one.

What Is Logic?

This is a question which we should have answered at the beginning of the book rather than now. We did give some kind of an answer, to be sure, but it was quite superficial. It was like trying to tell some one about gasoline engines who knew no physics. You have to tinker with the thing awhile and study it before you understand it. By this time we have done considerable studious tinkering with logic, and we ought to be in a much better position to answer this question.

Let me get at the answer by asking you a question. This is a cowardly method of handing out information which all teachers employ. *Why do you suppose the system of classes and the system of propositions have similar structures?* Because they both deal with *logical* entities, both are systems *of logic.* Classes and propositions are *logical* things, the connections between them are *logical* connections, and the manipulations to which they are subject are *logical* manipulations. In arithmetic we have *quantitative* things, *quantitative* connections, and *quantitative* manipulations. Geometry is a sort of mixture of

quantitative and *spatial* properties. Arithmetic is a pure system. And so is logic. If any one should ask me what I mean by "logic," I should refer him to the system of propositions and the system of classes and say: "Logic is a subject dealing with the things out of which these systems are made, of the ways in which they are interrelated, and of the ways in which they can be manipulated." "Disjunction" and "implication" and "included in" are purely logical considerations, for example. You will find them nowhere else but in the structure of our thinking. Logic is the study of the structure of thinking. I know no better answer.

Chapter Twelve

A PHILOSOPHY OF LIFE AS A DEDUCTIVE SYSTEM

LEWIS CARROLL was a rare specimen of humanity—a mathematician with a sense of humor. As a mathematician he was much interested in games and in logic. But the rarest thing about him was that his interest in structure, whether mathematical or logical, made a large contribution to the quality of his humor. There is not much humor in a game of chess or a syllogism, but Lewis Carroll could use either in a way that provided the most absorbing entertainment. Even the most rabid admirers of the famous Alice do not all realize that *Through the Looking Glass* is actually a chess game. Alice is a white pawn who moves through fields (white squares) and forests (black squares) and over little brooks (the divisions between the squares) and is made a queen when she reaches the eighth row. You can follow the game on a chess board if you wish: the author takes a humorist's license with the rules of the game, but that does not spoil the fun. The smaller number of those who have enjoyed Lewis Carroll's *What the Tortoise Said to Achilles* know the fun he could have with the ordinarily dull syllogism.

But very few Lewis Carroll fans know about the deductive system which he devised, called *The Dynamics of a Parti-cle.* It starts out with definitions of *Plain superficiality*, *acute anger*, *obtuse anger*, and so on. Among the postulates are statements

to the effect that a speaker may digress from any one point to any other point, that a finite argument may be produced to any extent in subsequent debates, that a controversy may be raised about any question, and at any distance from that question. A little later you find that particles are divided into "genius" and "speeches," that a surd is a radical whose meaning cannot be exactly ascertained. The whole thing develops into a beautiful piece of political satire and, if you are on familiar terms with deductive systems, is as much fun as the Mock Turtle's immortal story about his education from Reeling and Writhing, through Mystery, to Laughing and Grief.

How Important Are Deductive Systems to You and Me?

In recent years, since the introduction of the study of deductive systems to the subject-matter of logic, I doubt if there is a teacher in the field who has not had to face a withering barrage of hints from his students to the effect that their time might conceivably be spent on occupations more directly applicable to human life, such for example as investigating the question as to whether or not the moon is made of green cheese. The tyranny of the modern college being what it is, most students are polite about it. But sooner or later the teacher changes from offensive (take it in any sense that pleases you) to defensive tactics.

Mathematicians are in the same boat. How often since you left school have you had occasion to apply what you learned there about second degree equations, the properties of chords of circles, the square roots of imaginary numbers, the tangents and cotangents of angles? This may safely be regarded as a rhetorical question. How often do you expect to have to apply more than a half dozen of the theorems which we have cited

in the system of propositions and the system of classes? Again, I fear, a rhetorical question. Will you ever have occasion to remember that:

$$p . q . \supset r : \supset : p . \supset . q \supset r$$
$$a + b = a + (b \times -a)$$

Up to this point I am with the students. If they are interested in that which is directly useful they would save time by finding some kind soul who is willing to set down a concise summary of Useful Facts. In mathematics it would include the multiplication table, formulae for the areas of triangles and circles, etc. In logic it would include such principles as double-negation, non-contradiction, excluded middle, class inclusion, and perhaps a few more. This would leave the more intricate technicalities to the professional. And he is welcome!

When you buy an automobile you are presented with a handsome Handbook, full of pretty pictures and the simple rules about how to operate and take care of your new car. This handbook is fine as long as the car runs smoothly, but if you rely on it alone you are eventually going to have to call in the local garage man. He understands the car in a way which you never can with your little book. The same thing is true in other fields. No mathematician or logician in his right mind could agree that he is teaching only useful facts—and hold his job! He is trying to make you intelligent about a set of ideas which you are using all of the time. Numbers and plane figures and propositions and classes are used in more American Families than are automobiles. We all use them every day. And it is more than possible that a person who understands them by virtue of the fact that he has studied mathematics and logic beyond the point of immediate usefulness will use them more accurately than another person who has not. There is something important about knowing the more general prop-

erties of numbers and angles and propositions and classes. This is one answer to the objection of the short-sighted student. But, happily for the logician, he has another answer which is so much better that it seems unimportant to dwell on the first.

The Case of One Benedict Spinoza: A Philosophy of Life as a Deductive System

It is a long jump from the accomplishments of Lewis Carroll to those of a seventeenth century Jewish philosopher. But the two have at least this in common, that they both dealt in deductive systems. And they are alike immortal because each was supreme in his field, the former in writing logical nonsense and the latter in developing a rational philosophy of life.

No one opens Spinoza's masterwork, the *Ethics*, for the first time without getting a shock. It is written like a geometry! It seems that Spinoza, like Plato, had been much impressed with the type of knowledge produced in the field of mathematics. When an arithmetician talks about square roots, or a geometrician enlarges on the properties of isosceles triangles, no one is in a position to question his judgments. Why not? Because the mathematician *proves* his statements. He develops them carefully as parts of deductive systems. He bases them on assumptions which no one in his right mind would question, then works out the propositions step by step so that the whole structure has a firm foundation. If only philosophy could be like that, Spinoza thought. So he tried to *make* philosophy like that. He developed his philosophic judgments just as carefully as Euclid had developed his theorems. And when his *Ethics* was completed he believed that he had a philosophy of life as indubitable as Euclid's geometry. We know to-day that he was mistaken in this belief. But, after all, we know much more about deductive systems than Spinoza did. However, he

did something much more important than that without realizing he was doing it. That is often the way with philosophers.

It takes some courage on the part of the amateur, but if you will open the *Ethics* you will find that it starts right out with definitions of the words that are to be used in the system, Substance, Attribute, Mode, God, Cause of Itself, and so on. Then come the Axioms, just as in Euclid. "From a given determinate cause an effect necessarily follows"; "A true idea must agree with that of which it is the idea"; "The essence of that thing which can be conceived as not existing does not involve existence." Like other axioms, these might seem obvious until one sees what Spinoza can do with them as a foundation. Then come the propositions proving that besides God, no substance can be or can be conceived; that the mind does not know itself except in so far as it perceives the ideas of the affections of the body; that each thing, in so far as it is in itself, endeavors to preserve its own being; that it is impossible that man should not be a part of nature; that blessedness is not the reward of virtue, but is virtue itself, and so on.

In the words of Spinoza himself, the words with which he appropriately concludes his great work, "all noble things are as difficult as they are rare." The *Ethics* is both difficult and rare. Spinoza possessed one of the keenest intellects bestowed on any man. His book makes strenuous reading, but if you take the trouble to follow the steps in his reasoning it becomes as clear as a bell; the propositions follow upon the assumptions and one another as the night the day. Spinoza may have made mistakes, may have misused words, may have employed assumptions not stated at the beginning. It would not be appropriate to criticize the *Ethics* here. The *Ethics* is among the rarest of human achievement. Spinoza might have done as well if he had not bothered to follow Euclid's technique. He would surely have been more readable. But he would not have shown what

I think is all-important—that *a philosophy of life is a deductive system.*

You may not agree with Spinoza's conclusions. In fact, you probably do not. But that is beside the point. I am not urging the *Ethics* because it contains eternal truths. I am urging it because of its structure. It shows more clearly than anything else I know that whatever you think from the time you get up in the morning until the hour when you fall asleep at night, is part of a large system of thoughts. And that system is constructed, or should be constructed, just like a geometry or a system of propositions or a system of classes. Your thoughts are interconnected, all based on more fundamental ones, and the fundamental ones all lead back to profound assumptions which support everything else and which constitute the pattern of your life.

The Unique Importance to Us of the System of Propositions

And it is for this reason, primarily, that the study of deductive systems is significant. One of the explanations of why geometry has for so long been a part of the school curriculum is that for many centuries it has been the clearest example we had of a deductive system. It would be hopeless to try to defend the study of geometry in terms solely of its content. You concerned yourself with point and lines and planes in order to study the structure of thinking! This procedure was admittedly indirect, like taking up typewriting in order to be able to play the piano. Why not study the structure of thinking directly? The structure of the systems of propositions *is* the structure of thinking. You find out not only how propositions are interrelated, but you also see why they are interrelated.

You have probably noticed that in developing the system

of propositions you are actually working out the principles according to which the system is developed! Wheels within wheels. Eaton gives a fine example of this. You remember *Proposition 7*, which looked so easy yet had such a difficult proof? We wanted to prove that $p \supset p$. The interesting thing is that we already had at our disposal two statements which, when taken together, seem ample to provide the proof. (1) We knew from Theorem 6 that $p . \supset . p \vee p$ and, (2) we know from Rule I that $p \vee p . \supset . p$. Why are those two statements not enough to carry the point? p implies p v p, which in turn implies p, hence p implies p:

$$p . \supset . \boxed{\begin{array}{c} p \vee p \\ p \vee p \end{array}} . \supset . p$$

But *how do we know* that if one proposition (p) implies a second (p v p), and that second implies a third (p), that then the first (p) implies the third (p)? It is common sense to suppose that this is the case, and indeed we employ the principle very often. *But how do we know that it is valid?*

The answer is that we do not. Hence we cannot use it. If you study the system of propositions carefully you will discover that *no principle is employed in working out the system which is not either assumed at the beginning as one of the rules of the system or already proven.* As a matter of fact, the principle that if one proposition implies a second and that second implies a third, then the first implies the third, is stated thus as a later proposition:

$$\text{Theorem 19:}\quad p \supset q . q \supset r . \supset . p \supset r$$

and its proof depends on the proof of Theorem 7! Hence if we had tried to use Theorem 19 in proving Theorem 7 we should not only have been taking a step for which there was

no warrant, but we should also have been arguing in a complete circle. Proving Theorem 7 in terms of Theorem 19, and then Theorem 19 in terms of Theorem 7, would be like writing a check to remedy an overdrawn account. The check is no good because the account is no good, and the account is no good because the check is no good.

This is a digression, but it does show that in studying the system of propositions we are discovering not only the character of a deductive system but at the same time working out the rules according to which a deductive system operates. This is something like studying a microscope through a microscope. The system of propositions is in a position of unique importance. It is not only a deductive system, and hence worthy of study: it is also the deductive system which makes deductive systems possible, hence doubly worth study.

Terms, Defined and Undefined, in a Philosophy of Life

In an earlier chapter we likened the problem of working out a sound philosophy to that of mooring a boat securely. We said that the security of the boat depended on the paraphernalia of anchor and cable, most of which it is difficult to see because it is below the surface of the water. Now that we know more about the relations among propositions we possess a sort of inverted periscope which allows us to study more carefully what is below the surface. So instead of starting at the boat end of the problem we can now start at the anchor end. Instead of looking at a philosophic system by starting with our everyday judgments, the ones on the surface of our thinking, let us study it as we would study a geometry or the system of propositions, from its logical beginning.

Just one caution before we begin. Do not get the idea that

this is the way we actually do develop philosophies of life. Though philosophy is itself formal and should be studied formally, it grows informally. At first we notice only surface appearances. Later we take quick dives and swim for a few seconds under water. Still later as our skill develops we can stay under longer, open our eyes more, and look around. Only a few equip themselves with diving suits so that they can stay down and really study the underwater situation. This is the case with geometry, too. Our early knowledge in this field is quite informal; some of it fairly accurate, and the rest quite sketchy. It is not until we study geometry as the geometrician sees it that we really understand it. So now let us don our diving suits and go right to the bottom of that deductive system which we know familiarly as a philosophy of life.

In one of his books Joad speaks of what he calls the Three Dowagers of Philosophy: Truth, Beauty, and Goodness. If you add to these the idea of Reality, you have what are among the most fundamental concepts in philosophy. They are to philosophy what "point" and "line" and "plane" are to geometry; what "proposition" and "not" and "either . . . or" are to the system of propositions. Which of these are defined terms and which are undefined it would be difficult to say. Not wishing to embroil ourselves in a technical discussion of philosophic terminology, we shall make only the comparatively rough statement that each of these words has a common meaning, a meaning which is outside of the special meanings given in a particular philosophic system. For example, "truth" symbolizes a quality of coherence among ideas for some philosophers, while for others it expresses a correspondence of ideas to external objects. These are the meanings that the word is given *within two systems*. But the fact that both philosophers use the same word, "truth," is eloquent. It tells us that outside of the system there is a meaning for the word to which they both agree.

To choose another, for some philosophers "reality" is a spiritual unity, while for others it is a physical plurality. But, again, these meanings are given as the system is developed. They are, in a sense, the outcome of the system. But notice that both philosophers use the same word. We can compare this, perhaps, with the use of the word "line" in geometry. Lines on spheres and lines on planes are quite different, yet there is a common meaning which justifies the use of the same verbal symbol for both.

The Rules of a Philosophy of Life

You would not know much about a geometry if you knew only that it deals with points and lines and planes. There are countless things that can be done with these elements. In philosophy, too, the fun begins only when philosophers start making up rules for their systems. There is much pulling of hair and calling of uncomplimentary names: and the results are as different as euclidean space is from non-euclidean. You may be enough of an admirer of Keats to believe that beauty is truth and truth beauty, or you may go along with the pessimists and think that all truth is the extreme of ugliness. There are a thousand things you may think, a thousand thousand, and they all depend on the rules of the game as you see them.

The most basic rule of all is the one that tells you what philosophy is all about anyhow. A large number of philosophers think that the job of philosophy is that of studying and reporting upon an external reality. For them the cosmos comes first and truth is dependent on a relation of judgments to it. There are others, a smaller number, who would say that the job of the philosopher is that of asserting universal and necessary judgments. Hence many of these base their systems on the

Cartesian *Cogito ergo sum*, and wind up with a point of view that places the mind first and interprets the cosmos as the creation of thinking. Another way of saying this would be to state that the first group cites as one of its rules the Doctrine of External Relations, and that the second group plays according to the rule of the Doctrine of Internal Relations.

But already we are getting more technical than we should. Suffice it to note that when the philosopher has set down the rules according to which he is going to relate his concepts of reality and truth and beauty and goodness, then his system has been thoroughly characterized and fixed, as much as the type of space to be described in geometry is fixed by the set of axioms chosen as a basis for the later propositions.

Geometry is divided into books, one on triangles, one on circles, one on polygons, and so on. Philosophy is also divided into "books": Ethics, Aesthetics, Metaphysics, Epistemology, Politics, etc. When the philosopher comes to his book on beauty he will find that what he has to say will depend ultimately on the rules he set down earlier in the game. If he plays according to the Doctrine of External Relations, he will find the quality of beauty to be external to the artist. If he plays according to the Doctrine of Internal Relations, he will find beauty within the artist himself. The one philosopher will write a book on art which will be quite different from that of the other. And the same is true when they come to the book on problems of conduct.

This is not the place to set down a complete set of the rules of a philosophy of life. We mean only to suggest that there are such rules and to recommend that in studying various philosophic systems (idealism, materialism, pragmatism, naturalism, and so on), one search for the terms and rules on which they are based, as a way of understanding what they are trying to do. These are the anchors to which the boats are moored.

The Theorems in a Philosophy of Life

But a mooring would hardly be a mooring if it consisted only of an anchor. And a philosophy of life would not be a philosophy if we knew only its terms and the rules according to which they are related. It is what we *do* with the anchor, and what we *do* with the terms and rules, that is significant.

One of the things we noticed, at first to our bewilderment, was that the early theorems in a deductive system were so comprehensive that they seemed obvious. The reason for this is, of course, that the early theorems are broad generalizations in contrast to the later ones, which are more specific. The more we develop a system the more detailed our findings become. The analogy with building blocks is helpful in understanding this phenomenon. Every child knows that in building a tower the large blocks must be used first, and that the smallest blocks can most safely be placed near the top. The higher he builds his tower, the smaller the blocks he uses. Or you might use the analogy of a tree. Smaller branches grow out of larger ones. As you move toward the top of the tree the branches get smaller, and as you move toward the base the branches get larger. The reason for this is evident. The early propositions are basic to a large number of later ones. Propositions multiply with the rapidity of individuals in a genealogical tree. A lot of great-grandchildren follow their relationships back to the same pair of great-grandparents.

The same phenomenon is found in the organization of a philosophy. If you will accept this as but a rough characterization, I should be willing to say that the theorems in a philosophy may conveniently be divided into three groups: the broad generalizations which show the major trend of one's thinking, the still broad but more concretely significant principles which arise out of these generalizations, and, finally, the

more specific precepts which are applied to the immediate field of action. Let us consider each of these groups in turn.

Protagoras said that man is the measure of all things. This is as broad a generalization as a philosopher could make. Out of it comes a whole set of principles characterizing man's knowledge and indicating an attitude toward the conduct of life. It leads to those humanistic interpretations of reality which place man at the center of everything. About as different from this as anything could be is that famous early theorem in Spinoza's *Ethics* which states that besides God, no substance can either be or be conceived. This places man within a larger reality of which he is but a part. And it characterizes reality as monistic. In contrast to both Protagoras and Spinoza would be the theorem of Descartes to the effect that there are two fundamentally different and irreconcilable realities, mind and matter. Then there is another large group of philosophers who would assert at the beginning of their philosophy the statement that besides matter no substance can either be or can be conceived. This broad generalization leads immediately into all of the implications of a materialistic view of life. And in contrast to all of these there is the highly important theorem of the pragmatists which denies that we can know with certainty about reality, and therefore recommends that we settle all problems by considerations of their practical significances.

Even so hasty a survey as this shows that philosophers can be classified according to what type of theorem comes first. For many it is an attempt to characterize reality. Is it monistic, dualistic, pluralistic? Is it material, spiritual? Others, such as the modern idealists, think it more important to characterize knowledge first. What can men know, and how? What are the criteria of truth? For the first group a theory of knowledge refers back to theorems about reality. For the second, a theory of reality arises out of theorems about the nature of knowl-

edge. If you agree with a philosopher, his first theorems seem quite obvious; if you do not, it does not seem worth while to argue because argument leads back immediately to fundamental assumptions, and it is impossible to argue assumptions. They are assumptions and beyond argument.

The Second Group of Theorems

Once the general character of a philosophy has been determined there is a whole host of more concrete problems to be faced. There is inscribed on a building at Harvard University a famous question which would have been more appropriate at Radcliffe: "What is Man that Thou art mindful of him?" This is perhaps most prominent among the problems the philosopher wants to answer.

What is man? Is he a material object? Is he an immortal soul? Does he enjoy free will? Or is he completely determined in his actions? Descartes gave him free will. Spinoza refused it to him. The modern materialist describes him as a collection of blind electrical forces. The religious man endows him with a divine spark. Some say that he necessarily seeks bodily pleasure and comfort; others picture him as the seeker after eternal verities. Wordsworth pictured him coming into the world trailing clouds of glory. Matthew Arnold thought the management of children like a game of chess against the Devil. Rousseau saw him as a benevolent savage made evil by civilization. There are as many answers to the question as there are philosophies, and each answer refers back clearly and unhesitatingly to earlier and more fundamental propositions in the philosophic system.

There is another large set of problems, generally known as the cosmological ones, in which the philosopher also takes great interest. How describe the world in which man lives?

Or better, what is this reality of which man is a part? Among the most perplexing are questions concerning the character of space and time, thanks to the dilemmas which we have inherited from Zeno. Others concern themselves with the reality of such entities as numbers, ideas, and ideals. In general these cosmological problems are attempts to discover which aspects of our world are appearances and which are real. What shall we regard as superficial, and what as of fundamental significance? What place is there for God in our philosophic picture? Does he transcend the universe which he has created, or is he that universe itself? Or is God as much of a chimera as Santa Claus? Does the world exhibit purposefulness? If so, what is that purpose, that design? Or is the world's history made up of a blind concatenation of events?

It is difficult to treat adequately of this second group of theorems, if only for the reason that what are secondary problems to one philosopher are primary ones to another, and *vice versa*. We could state a whole series of problems about the problem of knowledge. Do we come into this life equipped with innate ideas? Or is all of our knowledge built up by a precise series of associations that are entirely the result of our sense experiences? Does the mind in itself contribute anything important to human knowledge? Are there any eternal truths? If so, how should we proceed to find them? If not, what is the problem of knowledge? Must man work entirely in terms of hypotheses tested and improved by experience? Some philosophers find it necessary to answer these questions before questions about man himself and about the cosmos can be asked. Some philosophers reverse this order.

But the important point is that in any one system these questions are all relentlessly interconnected. Whichever come first in your philosophy are most fundamental for you, and determine what you will think about the remaining problems,

as definitely as the early theorems in Euclid determine the character of the later ones, or the first theorems in the system of classes determine the ones that come after them.

The Third Group of Theorems

The specific precepts that have significance for human action are much more familiar than their earlier forebears. You will recognize them quickly. We are still below the surface but much closer to it, and hence we see them more clearly. The first group of theorems might be likened to the mooring rope as it is tied to the anchor: the second group would correspond to the middle section of the rope; and this third group is represented by the rope as it approaches the surface buoy.

"Eat, drink and be merry, for to-morrow we die." Here is a well-known outcome of certain earlier theorems about the nature of man and his cosmos. It is unnecessary to point out the earlier theorems on which such a precept is based. "Live not for this life but for the life that is to come" is a theorem with exactly the opposite meaning, and is based on earlier theorems from any entirely different philosophy of life. Rather than trying to cover a whole field of theorems of this third type, suppose we just list a number of the more prominent ones. You will find it instructive and entertaining to try to reconstruct the earlier theorems on which each is based:

Might is right.
It does not profit a man to gain the whole world if he lose his own soul.
Every man should cultivate and obey his inner conscience.
When in Rome do as the Romans do.
The end justifies the means.
A thing of beauty is a joy forever.
God's in His heaven; all's right with the world.
Man does not live by bread alone.

Thou shalt not kill.
That action is best which secures the greatest happiness for the greatest number.
Youth is wholly experimental.
Religion is the opiate of the people.
Humility and pity are as bad as hatred and pride.

More profitable still, as we shall see, is to take your own precepts for the conduct of life and try to find the theorems behind *them*.

From the list we have given it would seem that most of the theorems of this third group are ethical. This is not the case. We have quoted ethical ones chiefly because they are so familiar. There are equally many in the fields of art, politics, and science, which are significant for human action. We seek not only goodness, but also truth and beauty, and they are not necessarily the same, though they are interpreted in terms of the same fundamental theorems of the first group.

The Problems of Every Day

Now, at last, we break above the surface again. We emerge into a realm familiar to all of us. What are the questions we answer every day? Some of them seem quite trivial. When shall I get up in the morning? Shall I take a second helping of dessert? What shall I say to the peddler who has just rung the doorbell? Shall I take a short walk before supper? What shall I do this evening? As T. S. Eliot puts it in one of his poems: "Shall I part my hair behind? Shall I dare to eat a peach?"

There are others that have more importance. Whom shall I invite to dinner, or what girl shall I take to the Big Dance? Should I buy a new coat this winter, and if so what price and style? Where shall I go for my vacation? How shall I spend this extra ten dollars? Shall I give Her a birthday present?

And others with still greater significance. What newspaper shall I read? How shall I budget my money? Should I have a regular physical examination? Where shall I live? How shall I vote at this election? How much part should I take in community affairs? Shall I take up smoking? How much life insurance should I carry?

And, finally, the largest problems, ones which we face during crises in our lives. Shall I propose to this girl? Shall I knuckle down to the racketeer or call in the police? Shall I enlist in the army in time of war or be a conscientious objector? Shall I join a church? How shall I earn my living? What place shall I take in society? Shall my wife and I bring children into the world?

The important point is that there is no one of these problems, from the most trivial to the most momentous, that is not a part of a philosophy of life. There is no one of these surface problems that does not lead us below the surface again. If we would but recognize the fact, there is no one of these problems that is not part of a deductive system, and that system is the one most important to us because it is the one by which we live, the one in terms of which we decide what we shall do with the few years allotted to us on the terrestrial globe.

Consider, for example, one of the very trivial judgments. Shall I take a second helping of pudding? A second helping of *this* pudding on *this* particular day makes little difference, of course. But taking second helpings becomes a habit. It is part of the larger philosophy of second helpings, so to speak. I excuse myself by saying that the pudding tastes good. But, do I eat everything that tastes good? Of course not. Where do I stop? Why? Perhaps I have my figure to think of. Do I want to avoid corpulence? Which is more important, the pleasure of the taste buds or the pleasure of a lean body? Why? Perhaps the doctor has advised a diet. Which is more impor-

tant, a longer life on a diet not particularly pleasurable, or a shorter life in which I enjoy my meals? Why? Do I live to eat, or eat to live? We are immediately faced with an important problem of relative values.

One of the curious things that anyone can discover for himself is that if he will take *any* decision made on any day of his life and ask why he made it, and how he defends his reason, and how he defends the reason for his reason, and how he defends the reason for the reason for his reason, he will soon get back to a very profound philosophic principle upon which this action and many of his other actions are based. Try this procedure on *your* answer to any of the questions raised in the paragraphs above. Ask yourself Why? Why? Why? *Why* did you buy a new coat this winter? Because you want to be well dressed? *Why* do you want to be well dressed? Because Mrs. Jones is well dressed? *Why* do you want to keep up with Mrs. Jones? Or perhaps you dress well to help you in your job. *Why* is your job that important? Because you want to earn a certain amount of money? *Why* do you want to earn that much money? You want security and the maximum pleasures in life? *Why* do you want security and pleasure? There are countless thoughtful and wise men and women who find other things more important than security and pleasure. There is an increasing number who think that fashion is spinach. There are many respectable and interesting people who prefer old clothes to new ones.

To Be Logical Is to Live Fully and Happily

Nine people out of ten never realize that they are acting in terms of philosophy. They accept the superficial judgments of their neighbors (and the superficial habits that go with

them) without ever doing any thinking for themselves. They are swept along by the mob into which they happen to be born. Their lives are circumscribed by considerations of the next few hours: they do not look ahead to to-morrow. They accept the "common sense" values which may be grouped under the head of Popularity, the approval of the crowd.

I remember once seeing Raymond Duncan dressed in his Grecian garb in Grand Central Station, surrounded by a crowd of scoffers. I do not agree with his philosophy of clothes, but I know that he was head and shoulders above his scoffers if only because he had done some individual thinking and had dared to be different. Nine out of ten of us are like the sailor who moors his boat to a piece of drift wood, floating this way and that with every tide and wind. The tragedy, of course, is that these people live their lives and end by not having really lived them. To find the enduring satisfactions requires fore-thought and philosophic discipline. It is like learning to play an instrument. It takes application and is not much fun at first, but in the end it pays dividends. Philosophy pays dividends, it leads to the full life.

And then there are those who do dive below the surface, but do not stay down long enough to see the snarls in the mooring rope. For example, those who are good Christians on Sundays and bad ones on weekdays. Or those who do not go to church themselves yet send their children to Sunday school. This group understands that there are broad principles underlying their actions, but do not realize that these broad principles conflict with one another. They have been living according to a deductive system which conflicts with itself. They are like the poor geometrician who might at one time work according to the idea that vertical angles are equal and at another base his judgments on the idea that they are not equal. It does not work in geometry. It does not work in any

deductive system. And in a philosophy of life it is extremely messy. I find many students who in their tender-minded moments believe explicitly in the existence of the soul, yet who in their tough-minded hours in the laboratory will assert that only tangible objects are real. There are others who will admit that the use of force in settling an argument is wrong, yet who admit that they would support the next war. Question them a little, bring these conflicting propositions to light, and these people do plenty of squirming.

Like any deductive system that is valid, a philosophy of life must be consistent with itself at all times and in all of its phases. It just does not pay to scatter yourself in all directions. If you are seeking one set of values at home with your family, and another when you are away or at the office, you are bound sooner or later to be unhappy, profoundly unhappy. And not just because you may be found out. The danger of being found out is not the measure of a life. You will be unhappy because you will be in conflict with yourself. Shakespeare was not a student of deductive systems, but he managed to put into the mouth of Polonius the one highest lesson which they have for us: "To thine own self be true, and it must follow, as the night the day, thou canst not then be false to any man."

The Many Philosophies of Life

These are the reasons why the study of a deductive system as such, an understanding of its unique structure, is important. It helps more than anything else could toward the leading of the full life and the achievement of enduring values. But there is one more reason, and in some respects this is the most important of all.

You will remember that when we discussed geometry as a

system we found that there were a number of geometries, euclidean and non-euclidean, all valid and all leading to descriptions of unique types of special relations. We saw that there were at least two ways in which we could build a system of propositions, depending on how we wished to interpret the "either . . . or . . ." relation. We saw that there are a number of games that can be played with the same checkerboard. We mentioned several types of number system. And we realized that from among these possible systems we chose as the most important the one which was most significant in application. They were all perfectly good as deductive systems, but some were more useful than others.

And there is no need to tell you that there are many varieties of philosophy of life. How choose among *them?* At this point an extremely interesting and important question arises. *We cannot choose among philosophies of life without doing so in terms of a philosophy of life! To choose a philosophy of life is to make a philosophic judgment.* Any standards that you employ will be philosophic standards and hence beg the entire question. It is something like choosing the winner at a Beauty Parade. The judge may be partial to blondes, and you prefer brunettes. No winner of a Beauty Contest was ever *The Beauty,* she just fits the judge's idea of beauty. And, similarly, the philosophy of life which you choose will not be *The Philosophy,* it will be your idea of what philosophy should be.

It all goes back to the initial assumptions, or rules, with which you start. There are at least a half a dozen major sets of rules according to which you can play the game of life. When you study the history of philosophy you will find them all and can study them with care. But they are true assumptions. You cannot get behind them. There is no one, living or dead, who knows which is the best set. The only answer is tolerance of points of view that conflict with your own.

You can demand of a man that he know the general theorems on which his system is based, and you can insist that his theorems be consistent with one another, and you can ask that they all follow logically from the initial assumptions. On these points there is large room for argument and discussion and debate. You can show a man that if he is to act in his own best interests he must be logical, must organize his thinking into a valid deductive system. But you cannot question the rules according to which he plays his game, for there is no way of knowing that your rules are better than his, or as good, or half as good.

Until they get to this point the philosopher and the logician are apt to be exasperatingly smug. When they talk about the leading of a life they seem to be telling you what you should do and how you should act, where you are shortsighted and where you have become a slave to emotion. They seem confoundly self-righteous. But really they are not. They cannot afford to be. All they can show you is that effective thinking observes a type of structure. They can show you that you will be wise to follow this structure and not transgress its laws. But what you shall think within that structure is entirely up to you. We said at the beginning, when we were trying to characterize logic, that it divorced structure from content and concerned itself only with the former. There could be no better illustration of this than what we have been trying to say. Think what you will. No man may deny you that right and privilege. But think accurately. That the logician can and does ask.

PART FOUR

The Structure of Maturing Thought

Chapter Thirteen

THE MOVING PICTURE OF THINKING

EVERY sailor knows how inadequate a snapshot is to reproduce the full beauty of a sailboat. It catches the momentary and makes it interminable. It changes graceful motion to a static caricature of itself. The beauty of a sailboat is dynamic, and unless you are sailor enough to put motion into the snapshot, it can have almost no real meaning for you. The portrait painter faces the same problem. A good portrait is good for what it suggests rather than for what it is. A look in the eye may give a clue to a trait of character. A profile may suggest strength or weakness. But the subject of the portrait is a living thing and the oils on the canvas are dead. The more motion and vitality possessed by the subject of the snapshot or the oil painting the more grotesque its image will be. The boat is poised woodenly on the crest of a wave, or has sunk into a trough never to lift its magnificent bow again as long as we may gaze at the picture. There are personalities so vital that they defy the skill of the painter. Many things lose their meaning when their motion is frozen to nothingness. Try to get the thrill of a roller-coaster by sitting in a stationary car!

In this problem of capturing dynamic qualities the moving picture has been a great boon. A few seconds of "moving" images on a screen are often more satisfactory than a whole album of snapshots. Every winter the newsreel companies show

movies of storms photographed from bridge decks of ocean liners. These pictures make you grip the arms of your seat and hold your breath, give you that elevator feeling. They may even make you actively seasick! Do you remember the first movies you ever saw of wild animals? And, above all, do you remember the shock of the first moving picture you saw of yourself? The marvel of the moving picture is that events are brought to life with a vitality that is missing in the best of still photographs.

The Logician as Photographer

What has all this to do with logic and the structure of thinking? It is an admission of guilt and a confession of ignorance. Up to this point our study has been devoted to still-life pictures, snapshots of the structure of thinking. We have gone out with a camera that will take photographs of ideas and we have snapped them in various poses. Then we have taken the exposed film to the dark room and developed it. When we analyzed the resulting pictures we found a number of interesting things.

In the very beginning, for example, we found that pictures of the thinking that goes on in men's minds often reveal faulty structure. A large number of these faults in structure would have passed undetected if we had not had our camera handy to catch a "photograph" of the scene that would remain constant and fixed long enough for us to study it. Dealing logically with the utterances of the average politician is like taking pictures of a fakir who throws a rope into the air and climbs up to heaven on it. The fakir can fool the human eye, but he cannot fool the camera's eye. And, similarly, the politician can stir the emotions and mislead his audience, but he cannot fool the man who will take out pencil and paper and make an impersonal analysis of what is said.

We took pictures of syllogisms in various poses, in their native haunts and in the captivity of the textbook. On examination these pictures showed clearly that all syllogisms exhibit the same fundamental structure. We took pictures of different types of argument; disjunctive, implicative, dilemmas, etc. All of these pictures were helpful. We went into the laboratory and took pictures of scientists at work and showed the ways in which their complexities of data are gathered together into generalizations. We took snapshots of the ideas in detective stories. We got some good shots of ideas in the act of converging on hypotheses. And, finally, we put all of our snapshots into an album so that we could study them as a group. When we did so we found that they all showed evidence of a kind of structure which we called the structure of a deductive system. Some of the later pictures were pictures of games. Others were pictures of propositional and class systems. Yet all were alike in that they were pictures of deductive systems. Some of these pictures, for example that of the syllogism, were so intricate in detail that we had to enlarge them before we could appreciate the complexity of their organization.

It is a matter of history that the first attempt at a moving picture arose out of a dispute as to whether or not all four feet of a trotting horse leave the ground at once. Cameras were lined up along the track, and strings were stretched across it at intervals so that the horse in trotting would break them and release successive shutters. When the whole series of pictures had been developed they were examined, and the question settled. The horse *did* leave the ground. Pictures often show us events that occur too quickly to be caught by the human eye. A camera will show a bullet leaving the muzzle of a gun. And so, too, the camera of logic will show details that would not appear in the ordinary reading of a book or hearing of a speech. Cameras are far more reliable than eye-witnesses in re-

porting events which quicken the emotions. The famous pictures of the coronation of George VI give many details missed by eye-witnesses, though surely the event was slow enough in unfolding. And the camera of logic gives a similar victory over other frailties of human nature.

The Botany of Logic: Glass Flowers and Japanese Flowers

But now for the admission of guilt. We have gone out to take pictures of ideas, but unfortunately we have equipped ourselves only with still cameras! Perhaps you have suspected all along that something was lacking. Most students of logic do. One needs more than a still camera to take pictures of all of the structures of thinking. If some one came along holding in his hand the most intricate and skilfully constructed glass flower you had ever seen you would admire the workmanship and find a certain fascination in it, but if he raised the glass flower and asked you to smell it you would not accept the invitation. A glass flower is only by kindness and by similarity called a "flower." It has all of the structure of a flower at one instant of its life history. But it is not a flower.

When a logician comes along and asks you to "smell" the intricate model which he has made of the structure of deductive systems you will not be interested in the proposal. We hope you are fascinated by the model, and that you admire the workmanship, but we would be willing to lay a heavy bet on the proposition that you have not been fooled into believing that the structure which we have analyzed and discussed in all of its various details is alive. Of course it is not. Perhaps you have wondered why we study such a dodo as a syllogism at all. In everyday life people do not think in syllogisms. Dilemmas are rare birds, too, scarcely ever encountered outside of formal debates. Our

thinking does, to be sure, organize itself at any one moment into a deductive system. We want to know if its structure will safely bridge the gap between premises and conclusions. But thinking, like a living flower and unlike a bridge, is *growing*. The word we use when we want to speak of the growth of thought is "maturing." *Thinking matures*. And just as no snapshot or glass model will show the growth of a flower, so no deductive analysis of thinking will reveal the structure of the maturing of thought.

Look back over the ground we have covered. We have seen that the nature of the theorems developed in any deductive system is entirely dependent on the initial assumptions upon which that particular system is based. The development of the system is the working out in all of its details of the implications of the material with which we start. You cannot make a silk purse out of a sow's ear. And, similarly, you cannot get a spherical geometry out of the axioms and postulates of plane geometry. In building a deductive system you never end with more than you had in the beginning. The syllogism, for another example, takes you right around in a circle. Its conclusion simply makes explicit a statement implicit in the premises. It is truly said that there is nothing logically new in a deductive argument. The newness is entirely *psychological*.

Since we have been talking about flowers, we may well cite the interesting case of the little artificial flowers for which the Japanese are so famous, flowers that expand or "blossom" when placed in water. We are delighted when we see them unfold, but we know that they are not alive, that nothing appears during the unfolding except what was already there compressed into a very small space. The flower does not really grow. A deductive system is like one of these "flowers": it expands, but it does not grow. When Euclid worked out his geometry he was simply expanding a set of fundamental postulates, and he

could get nothing in his theorems but what was already contained in the postulates.

In the early days of philosophy men believed that there was a single set of fundamental postulates to which all would agree and which would form the basis of the one true philosophy of life. The working out of one's philosophy was simply and solely a problem in deduction. In the Middle Ages men argued endlessly about theorems that might be deduced from the postulates authorized by the Church. How many angels could stand on the point of a needle? If there had been life insurance in their day, they would have wondered if Lazarus could have collected a policy when he rose from the dead! To-day we recognize the efforts of these thinkers as narrow. They never questioned their fundamental assumptions. How could their philosophic thinking grow? The answer is that it could not. But any undergraduate to-day can tell his logic teacher that thinking does grow. It is not like either the marvelous glass flower or the Japanese flower. It is like a real flower. It is living and vital.

Logic for Undergraduates

One of the interesting things about the kitten who shares my family life is that he is beginning to go out nights. He is getting that self-reliance, that sense of adventure, that cosmopolitan air. Not long ago his world was bounded by the four walls of the house. A few years hence he will be content to rest rheumatic old bones before an open fire and sleep away his old age. But now he is going places and doing things. At the height of his roly-poly clownishness the whole family wished he might stay that way forever. At present we are looking forward desperately to the time when he will not philander audibly and nocturnally under the bedroom window. Unfortunately he did

not remain a care-free kitten and, thank biology, he will not always play Lothario. He would be something more than a cat if he did either of these things.

One of the interesting things about college students is that they are beginning to go out nights—intellectually I mean now. Children are children, and they are so much fun that we often wish to stop the clock and enjoy them forever. Inevitably they come to the physical awkwardness of the high school period, achieve maturity and, like our cat, end their days before the open fire. But intellectually it is often far otherwise. They arrive at college with a complete set of ideas—some one else's ideas. Their confidence in this heritage is extraordinary. They know all there is to know about God except his latitude and longitude and the color of his beard. They understand perfectly the good life and are prepared to follow it with all of the smugness of immaturity. Some never go beyond this stage: their entire college course is spent in bolstering the ideas they brought to college with them. Is the fault theirs?

There is a modern theory of education which asserts that the teacher is responsible for the entire success or failure of the educative process. It is not true, but it is healthy, especially with reference to the logic teacher. The fault, however, is not so much with the teacher as with what he teaches. His pupils know it: his colleagues know it; he alone remains oblivious. He is absorbed in his subject, like a man playing chess. And there is no man so absorbed as an absorbed logician. Every logic teacher will remember the tolerant indifference (to put it mildly) on the countenance of his class at the mention of syllogisms and dilemmas. And every logic teacher gives up telling his students that this is the way people think. The students know better.

We treat students of logic as if they were in their dotage. We dust off a comfortable easy chair in front of the fire and invite them to sit down. THEY DO NOT WANT TO SIT DOWN.

They are young. They want to go out nights. The point is so obvious that it is startling. There is no aspect of deductive logic in which the conclusions contain more than was given in the premises. Deduction does not get anywhere. But people, and particularly undergraduates, *do* get somewhere in their thinking. Their thinking grows and matures. They move on from one deductive system to another, not selecting each by chance as one would take packages out of a grab-bag, but achieving a more and more intelligent point of view. Old ideas are discarded, new ideas come over the horizon of understanding. Thinking is a vital and active process.

Have you ever kept a diary? Do you remember looking back over it and realizing how silly your ideas of a few years ago were? The day the senior is handed back his batch of freshman themes is particularly notable for this experience of mingled shame and superiority. Some one once said that every doctor of philosophy who is worth his Ph.D. lives to find his doctor's thesis immature. You see, this process of intellectual growth never ends. Do you remember when you thought a brass band the pinnacle of musical accomplishment? Do you remember finding in the closet at home a book which was your favorite years ago, and wondering how you could ever have liked it so much? These are humble illustrations of the growth-dimension in our thinking. We never stay within the bounds of a single system of ideas. We are always changing our fundamental assumptions. The great difference between intellectual growth and physical growth is that the latter comes to a definite end while the former does not. There is no end to the search for ideas and understanding. We all know elderly people who are still growing, still learning, still maturing in their minds. These are the old people we admire. They are rare, but they show the possibilities. Why does not logic talk more about this aspect of thinking? Is there no structure of mental development?

Every reader of this book has probably found his interest in it increase decidedly when he came to the material on induction. It certainly approaches what actually takes place when a person thinks. It is more vital and practical than deductive logic. Build hypotheses and test them in experience. This is what we are doing all the time. It was not long after the Middle Ages that that colorful but corrupt Elizabethan courtier, Francis Bacon, saw the possibilities of the inductive approach. Do not start with basic assumptions and work down deductively from them. Be vital, cherish the open mind. Start with experience and build up hypothetical generalizations. That, for Bacon, was the technique of maturing thought.

But logic has moved a long distance since his time. Now we know that induction does not live up to this promise. We know that it is applicable only to certain aspects of knowledge. It is important in telling us about our natural environment. Its hypothetical character gives it vitality, for it recommends the open mind and suggests the possibility of better understanding ahead. It deals with problems of utility, of the successful manipulation of the environment. But what about problems of value? Usefulness is useful only to some end. It is a common observation to-day that our understanding of how to do things has outrun our understanding of what to do. Induction has never given a value judgment. Yet we live our lives in terms of values, and students of logic are in a period of intellectual growth when value judgments are running wild in their heads. If there is a structure to maturing thought, we must seek it elsewhere.

Why We Have Not Studied the Dynamics of Thinking

We are like a child confronted on the Fourth of July by the wonders of a whirling pinwheel. Its noise and motion scare him

and its fiery pattern seems to be sheer magic. He will boldly approach and examine it when it is nailed too tight and does not whirl, but once it starts wheeling he jumps away. Similarly, the pattern of developing thought is given a high magical significance, and when young men and women start lurching along its path we are a little frightened by the possibilities. Nail the student tightly in front of the professional desk to take down as gospel all he hears and it is easy to give him the deductive logic necessary to support preconceptions. But he will pull and strain, and when he breaks away we jump back because we do not understand what is happening. Things seem to have gone beyond our control. Anything may happen! The child does not understand the pinwheel because he is afraid to go near it. The teacher does not understand vital and developing thinking because he does not often enough have the valor to encourage it.

A famous educator is reported to have said that if he were founding a college he would first build a dormitory. If there were a little money left over he would construct a library. And then if there were funds still at his disposal he would erect classrooms. There is something important here. Our minds are most in motion in informal moments, in bull-sessions, when absorbed in books—and often least in motion in our classrooms. We must try to capture the mind's vitality. Perhaps we can remember some of the intellectually exciting moments in our lives, those moments indelibly marked in the memory: the day we first read Thoreau's *Walden*, our first encounter with a consistent and rabid atheist, the day we were introduced to a social reformer.

Another reason why we have neglected the study of the structure of maturing thought is that anything in motion is difficult to analyze. To return to an earlier metaphor, we have not yet devised a camera that will take accurate pictures of the mind *in motion*. Logically we are at the stage at which pho-

tography was when they were lining up cameras along the race track to find out what happened when a horse trotted. What happens when a mind thinks? One of the curious things about the mind is that while we are conscious of our thinking and seem intimately acquainted with it, as soon as we try to examine it the only tool at our disposal for probing is the mind itself. We are like a far-sighted person trying to examine his eyeglasses by removing them. He can no longer see clearly. When the mind is occupied with the examination of thought in motion, the thinking which it wishes to observe has stopped! Maybe the structure of maturing thought is as much a product of the imagination as the proverbial pot of gold at the rainbow's end. Perhaps there is no structure of mental development. We had best try to settle this question first of all.

Two Examples of Maturing Thought

What do we mean by maturing thought? We mean a series of changes in philosophic point of view. We mean a series of changes from inadequate judgments to more adequate ones.

When you were very young you probably pictured God as sitting in a golden chair up in the sky just over the rim of a particularly substantial looking cloud, where he could look down on us all and see everything we did. He wore a beard, looked thoroughly fierce, and carried some efficient tool of punishment in one hand while he shaded his eyes with the other and peered down directly at you. You may have gone through another period when you could not believe in the existence of God at all. You had come to see certain difficulties in the earlier concept and now could work out no picture that would have a place for a being who did not have physical existence yet could see everything that went on, who knew all there was to know yet busied himself listening to human pray-

ers, who was all-powerful and benevolent yet allowed a terrible automobile accident to snuff out the life of a friend who was as innocent of wrong-doing as any one else on your street. Perhaps you came later to reaffirm your belief in God, but now a very different kind of deity than the one pictured in your childhood thoughts. Now he might be the symbol of a moral order, or perhaps a necessary idea in the mind of man. Your thinking about God had become less and less childish, more and more mature. Of course this is not the only way in which your thinking about God might have matured, as we shall see, but it will serve as an illustration.

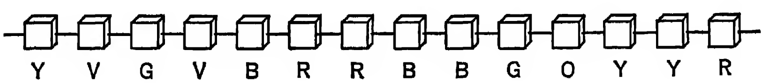
Let us take another, this time from the field of the arts. You probably do not remember the day when you first realized that music was fun to listen to, but you may recall getting a tremendous thrill out of a march. That was *great* music. It was the only great music. Then a time came when you were a little more discerning and marches began to pall; after all they were fairly obvious things and all much alike. You still liked the dramatic in music, but now the drama had to be more sophisticated. Suddenly you began to take notice of a fellow named Wagner. You sighed to the evening star and galloped through flames to save your lover. It was an important day when you learned of the existence of symphonies. Tschaikowsky had all of the requisite noise and excitement and drama, and here it was woven into an interesting pattern. You began to notice themes developing and recurring. This was a new idea in music. Here was music for its own sake, not just for marching or acting. Sometimes you got a love song on the French horn, sometimes a Cossack march. But it was symphonic. Later you found Brahms, and he raised you to heights of ecstasy and emotion hitherto unsuspected.

Suddenly you realized that you had come a long way since that day when the joy of joys was to hear something by John

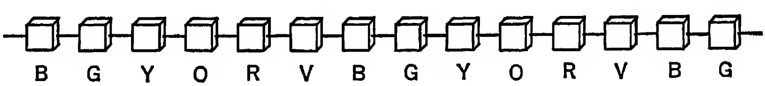
Phillip Sousa. Sousa seemed childish now. How you had come on! Then there was the later day when you first heard Beethoven's Ninth Symphony. You were convinced that this was the peak in all music. Nothing could touch it. It had everything of form and color that one could ask. Then you took up the cello, and began to get on the inside of this business of making music. That was when you discovered Mozart, who in your earlier days had seemed trivial and tinkly. There are lots of ways of maturing in music, too. This one will give you an idea of what we are seeking. But has this process structure?

The Structure of Maturing Thought Is Not Arbitrary

We mean, of course, does it have a meaningful structure? There is nothing that does not have structure, but some structures are without significance. If you had a box of beads and each bead were tinted with one of the colors of the rainbow, you could make a necklace by taking beads at random and putting them on a string. You might, then, get something like this:



The structure represented here has no meaning. No one color has a relationship to the color next it on either side except as the result of accident. You might, on the other hand, take the rainbow as a model and select the colors of your beads in the order in which they appear in it. Your result in this case would be:



This structure has a design. Each green bead, for example, is found between a yellow and a blue one, and there is a relationship between these colors which all artists recognize. A certain pattern is at work in this second case.

Apply this idea to the structures we have found to exist in deductive arguments. Let white beads stand for judgments, and black beads for the relation of "therefore" between judgments. If we select judgments blindly and string them into an argument we would get an arbitrary structure, such as the following:

This would be a horrible necklace! No, we may not string these beads in any manner that pleases us at the moment. The important point we discovered in studying deduction was that different types of judgments had to be strung in certain definite relationships along the string of argument if the result was to be valid. Again, pattern was at work.

Is there a pattern at work in the process of maturing thought? Is there a structure that must be obeyed? In the course of our intellectual development we are stringing the beads of our various judgments. May we string them in any fashion? Do we choose our ideas arbitrarily? At first it may seem that we do. There are many who will take exception to the illustrations given above of the development of the concept of God. There are other paths of approach to the idea of God, and other mature concepts of deity. And there is the same latitude in musical taste. In other words, there are many different mature and well considered philosophies of religion. And among mature musicians there are many disagreements about the greatness of composers. Some will put Wagner at the top, some Bach, and there are defenders of jazz whose competence to

judge maturely may not be questioned. Obviously these beads can be strung in a number of ways. There seems little hope of finding a logic in this process.

But there is one essential point that must not be allowed to escape us. After all, the phrase "maturity of thought" has significance. It would be safe to say that we know mature thinking when we encounter it, or that we *should* recognize it. As a matter of fact it is the immature who do not recognize maturity when they meet it. We should be suspicious of the Democrat who asserts that to be a Republican is to be immature in one's political thinking. There are mature Daughters of the American Revolution, mature communists, mature priests, and mature atheists. The man who drifts aimlessly from one doctrine to another, changing his allegiance with the last argument he encounters, or the last book he reads, is not mature. The development of a mature understanding is a process that has continuity, and that continuity involves a pattern.

Consider a familiar example. One of my students comes to me with a worried look and says that he can no longer believe in God. Why the worry? Does it not signify that there is a necessity in this step which is beyond his control? Otherwise why not just keep on believing in God and forget one's troubles? But we cannot do that. Whenever we take a step forward that step is in a sense taken for us. To return to my student, *he* can no longer believe in God. That is the point. He would be the last to testify that he has an arbitrary and whimsical control over this new judgment, to think it or not to think it. His troubled countenance is ample evidence against such a contention. He has had a new experience, or put together old ideas in a new way, and he suddenly finds that he can no longer believe something that he has accepted for years.

We all have this type of experience. Do you remember the day you found yourself sitting down for the first time to read

poetry because you enjoyed it? Do you remember the day you first heard your father swear a blue streak? Or the time you discovered that girls are interesting, and not just a nuisance? Or the day some one told you about Darwin and the theory of evolution? For those further along there is the day the first child is born. Somewhat later comes the fiftieth birthday. These are times of new intellectual perspectives, times when our thinking accelerates and sweeps us ahead.

The Personal Equation and the Difficulties Which It Introduces

But for all of us these moments of acceleration are different. We do not all mature, but those who do, mature at different speeds and in different directions. This is what makes the moving picture of maturing thought so difficult to get. The only thing to do is to take as many pictures as we can, disregard the subject-matter of our results, and see if we can find a structure common to all movies of maturing thought.

What happens when thought matures? This is very like the question, What happens when a horse trots? With regard to trotting it is the movement which interests us. We do not care about the particular horse that is trotting, or the length of the track he is racing on, or where the track is located, or the name of the man driving the horse. We are interested in a picture of *trotting*, a picture of a process. The same is true in taking a "picture" of the process of maturing intellectually. It does not matter who is maturing. And it does not matter what is stimulating this process; what experience, or teacher, or book. We are not interested in what, specifically, our subject thought yesterday, or the day before yesterday, or last year; no more are we interested, specifically, in what he thinks to-day, or what he will think to-morrow, or next year. We want to find the

structure of the process in *any* individual, one that will accurately describe *any* advance in maturity.

The great difficulty is that the process, wherever we find it, is so personal. A man's thinking in its development is bound up with his history and seems to be meaningful only in relation to him. His intellectual growth is definitely his own in a sense in which a syllogism cannot claim to belong to any one. The famous syllogism about Socrates' mortality has no personal or individual stamp. It belongs to every one. A man's intellectual growth is his own in the sense in which his hand is his own. It is definitely *his hand*. But he does not control its growth or its shape. The thinker experiences an impulse toward development, an impulse toward what is for him a novel synthesis of ideas.

It is profoundly true that no man can think whatever he pleases. Ideas have a directed force. Recognition of this force, and obedience to it, are signs of the open mind. Man has this control over his thinking, that he can close his mind to new paths, become intellectually static and dogmatic, decide to rest comfortably in one set of ideas rather than experiencing the discomfort of rising to his intellectual feet and moving ahead. But when this happens he ceases to think, in the most fundamental sense of that word. He has grown mentally decrepit. Man controls ideas in that he can encourage or discourage their development: ideas control man in that the direction of their development is beyond his command.

There is no parent or teacher who, having with effort and excitement and pleasure worked out a mature philosophy of living, is not tempted to impose this on children or students. This is human nature. How will I feel if my daughter does not come to love Mozart as much as I do? What would be my attitude toward a son who voted the Republican ticket? Am I not going to dismiss as superficial the person who has never

discovered the philosophy of old clothes? Yes, but logic moves against human nature. We have seen that it is human nature to mix emotion with reason, and that only a careful regard for structure keeps us from the consequent pitfalls. It is also human nature to believe that the philosophy of living you have worked out is the best philosophy, and that any other point of view is less mature. Of course you think you are right. If you did not, you would not think as you do.

But what about the other fellow, whose mind is as good as yours, who has done the same careful thinking and arrived at quite a different point of view? The logic of maturing thought has a lot to teach us about this other fellow and about our intellectual relation to him. But first we must acquire some notion of the structure involved. In the following chapter we shall see what can be done in this direction.

Chapter Fourteen

HOW THINKING MATURES

MEN are like lobsters. This is not a reference to the fact that both get red when placed in hot water. There is a more fundamental similarity. A lobster has what would to us be a highly inconvenient method of physical growth. His body is encased in an armor of hard shell, jointed very much like the armor of the old knights. And it is like the armor of the old knights in another respect. It does not change size. I suppose the lobster does for himself what every economical mother learns to do for her children. He provides himself with a suit of armor somewhat larger than his body needs, and continues to wear it after it has begun to be tight in places. But, unlike the mother, he cannot just go out and buy another outfit when the old one is outgrown. The only method at his disposal is to cast off the old and go into hiding until he has grown himself the protection of a new suit.

This must be inconvenient. Suppose our skeletons had to be discarded every year or so when we were in the growing stages, and that during the periods while we were acquiring new ones we had to stay in bed, jelly-like masses of flesh unable to support themselves. Or, to come closer to the situation of the poor lobster, suppose we grew all of our clothes and had to wait stark naked after the old outfit was outworn and until we could develop a brand new cut of cloth. "Where is little Willie?"

"Why, he is in the attic, you know. His new clothes will not be ready for three weeks!"

Fortunately our bodies grow. Our skeletons get larger; our skin keeps up with the skeleton and provides us with a continuously good fit on the surface of the body. But with ideas it is different. Ideas do not grow, they change. We change our ideas the way the lobster changes his shell. We have some disillusioning experience and find ourselves saying: "I am going to have to give up the idea that men should resort to violence to settle disputes." We *give up* the old idea, that is the point. And when new significances take possession of our minds we may say: "I've just got hold of a new idea. Private property is the source of our social troubles." This idea is a *new* idea. It takes the place of another which we now find inadequate. Ideas in themselves do not change: they are what they are. They act according to a law similar to the physical law of displacement: the logician calls it, as we know already, the Principle of Non-Contradiction. During the process of maturing, new ideas come in and push out old ones in much the same way that discs in shuffleboard push one another out of the scoring area.

But the change is not as sudden as the crash in shuffleboard. New ideas do not immediately take the places of older ones. There is an intermediate period of intellectual uncertainty when we do not know exactly what we think about the problem in hand. You might say that the building of an idea is like the building of a house. Suppose you are living in a house that has become too small for your needs and you want to erect a newer and more modern one on the same site. Before this can be accomplished the old house has to be torn down. This takes time. It also takes time to erect the new structure once the ground has been cleared. During this interim you and your family have to camp out, so to speak. Every one who has gone

through this experience knows that it is uncomfortable and inconvenient. But if we have courage and foresight we can look ahead to the joys of the new.

I am not enough of a biologist to know what would happen if our lobster lacked the courage to shed his outgrown armor. But we do know what happens when men lack the courage to shed old ideas. Their knowledge stops growing: they stagnate: they become narrow-minded. But worse than this, they get so accustomed to the old ideas and so confident in them that they become arrogant and belligerent. Somehow the longer we hold an idea the more it dominates our personality, until in the end it becomes an obsession. "What was good enough for your father is good enough for you!" And, worst of all, those who have stopped maturing intellectually resent bitterly the intrusion of more mature minds. This resentment will run so rabid that they soon become indignant at the intrusions of any who disagree with them. We all know that the results may be tragic.

Socrates and the Citizens of Athens

There is no story more tragic than that of the trial and death of Socrates. He was tried on the charge of having corrupted the youth of Athens, and was condemned to drink the fatal hemlock. We all know the story. The oracle at Delphi had said that no man was wiser than Socrates. Certainly no one was more surprised than Socrates himself at this pronouncement. And he immediately set out to refute the oracle by finding a wiser man.

In his speech of defense at the trial he tells what happened: "Accordingly I went to one who had the reputation of wisdom, and observed him—his name I need not mention; he was a politician whom I selected for examination—and the result was as follows: When I began to talk with him, I could not help

thinking that he was not really wise, although he was thought wise by many, and still wiser by himself; and thereupon I tried to explain to him that he thought himself wise, but was not really wise; and the consequence was that he hated me, and his enmity was shared by several who were present and heard me. So I left him, saying to myself, as I went away: Well, although I do not suppose that either of us knows anything really beautiful and good, I am better off than he is—for he knows nothing and thinks that he knows; I neither know nor think that I know. In this latter particular, then, I seem to have slightly the advantage of him. Then I went to another who had still higher pretensions to wisdom, and my conclusion was exactly the same. Whereupon I made another enemy of him, and of many others besides him."[1] And so Socrates went on, to poets and artisans, and all who pretended wisdom, and always the result was the same. One can see the picture clearly, and imagine how careful some must have been to avoid the inquisition of this man.

Then suddenly Socrates realized what the oracle had meant. "And I am called wise, for my hearers always imagine that I myself possess the wisdom which I find wanting in others: but the truth is, O men of Athens, that God only is wise; and by his answer he intends to show that the wisdom of men is worth little or nothing; he is not speaking of Socrates, he is only using my name by way of illustration, as if he said, He, O men, is the wisest, who, like Socrates, knows that his wisdom is in truth worth nothing." And so Socrates went about, the self-styled gadfly of the state, questioning and questioning, raising doubts continually, and he left in his path men who hated him and spread lies about him. In the end he was placed on trial for his life, accused of corrupting the youth with his questions.

[1] This and the two following quotations are taken from pp. 113-4, 115, 126, resp. of Vol. II of *The Dialogues of Plato* (Oxford), translated by Jowett.

Did he corrupt the youth? Socrates answers this charge directly in words that will never be forgotten as long as men seek knowledge and understanding. "... If any one likes to come and hear me while I am pursuing my mission, whether he be young or old, he is not excluded. Nor do I converse only with those who pay; but any one, whether he be rich or poor, may ask and answer me and listen to my words; and whether he turns out to be a bad man or a good man, neither result can be justly imputed to me; for I never taught or professed to teach him anything." How could a man teach anything simply by asking questions? Socrates sometimes called himself the intellectual midwife. He assisted at the birth of other people's ideas. He was no more responsible for the ideas he brought into the world than the obstetrician is for the tantrums of Mary-Jane. Socrates was put to death not because he corrupted the youth, but because he questioned persistently and raised doubts. People do not like to be questioned, to have doubts raised. They are comfortable in the familiar old ideas; they do not want to be bothered by having to seek new ones. But wisdom is not acquired comfortably. In the realm of his own thinking every man is a pioneer. And the life of a pioneer is hard.

The Importance of a Willingness to Doubt

Epictetus once said of the philosopher what might be said of all men: "What is the first business of one who practices philosophy? To part with self-conceit. For it is impossible for any one to begin to learn what he thinks he knows already." Socrates taught the importance of doubting. Willingness to doubt is the first prerequisite of mature thinking. One cannot entertain new ideas until the older ones have been removed any more than our lobster can grow a new shell until the old one has been discarded. Doubting is a painful process, but it is

as necessary to maturity as the dentist's drill nowadays to usable teeth.

Perhaps the youth of Athens would have been better off without the disturbing influence of Socrates. Why let the gadfly bite us? Why allow ourselves to be questioned? Why break down ideas? Why not leave people alone? Why seek maturity of thought? To ask these questions is like asking why seek beauty, or why lead the good life. Beauty and goodness are their own excuses as values. And so is wisdom. We seek wisdom because by its very nature it is better than ignorance. To be ignorant is to have false ideas. To have false ideas is to think things are so that are not so. No man intentionally harbors false ideas. Ignorance is not bliss, it is incompetence.

There are none to whom this idea is more important than it is to undergraduates. Santayana speaks in one of his books of what he calls "the chastity of the intellect," that state of suspended judgment in which we question our own ideas, entertain other ideas from all possible intellectual sources, and refuse to commit ourselves. When we get out into life we have to commit ourselves, we have to marry ourselves to one philosophy or another; but in college we are protected from life, presented with four years in which we may think profoundly and widely without acting. Some say that this is the weakness of college; it is too protected. But quite the contrary, its protection and its chastity are the college's great virtues. Students are given the opportunity that will never come again of having time to think, of having access in the classroom and the library to the greatest thoughts that are recorded in our intellectual history. "Behold," says Epictetus, "behold the beginning of philosophy!—a recognition of the conflict between the opinions of men." For there are all kinds and varieties of mature philosophies of life in conflict with one another.

Must we all go through the painful process of doubting and

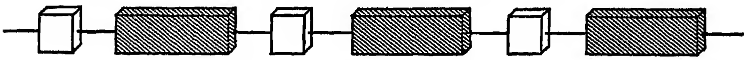
questioning? Is it not true that for some, and many of the citizens of Athens might have been included among them, doubting leaves an emptiness which is never filled? If you encourage a man to tear down his intellectual house, can he always build another? Not always. But I am willing to assert categorically that he who cannot undergo profitably the doubting process will never achieve a mature understanding. I have encountered students who are miserable when not told what to think, confused and bewildered when forced to read contradictory arguments, who think college is a place in which they will learn all of the answers. If they cannot rise above this childishness *they do not belong in college.* Our colleges are places for intellectual growth, and there is no growth without doubting and the willingness to discard outworn ideas.

The importance of thinking, of maturing, on the part of the intelligent individual cannot be overestimated. The individual must think for himself. We dare not risk taking our ideas from parents or teachers or any one else. Where did they get *their* ideas? From *their* parents and teachers? Somewhere the series must end. Some one must have done some thinking at sometime, and if we are to progress intellectually men must always insist on doing their own. The intellectual authorities under which you are born are purely accidental. Are you a Christian? You were born in a Christian country. Do you believe in rugged individualism? Your father owns his business. Are you willing to believe what you are told? What you are told is an accident of the time and place of your birth. And people are told so many different things. Competent authorities disagree in every field from religion to politics. This would be a stuffy and an ignorant world if people did not insist on doing their own thinking. Thinking is profitable and to raise doubts stimulates thinking.

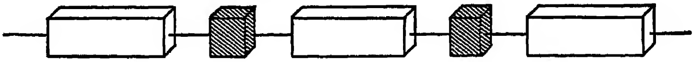
The First Principle in the Logic of Maturing Thought: Beads on a String

For all our digressions, we have made progress toward the understanding of the structure of maturing thought. Having considered lobsters and shuffleboard discs, we now possess a sketchy likeness of our new subject. We entertain a certain concept, discover that it is doubtful. A period of uncertainty elapses, then a new and more adequate idea is entertained. The closeness of this to the analogy of beads on a string is at least suggestive.

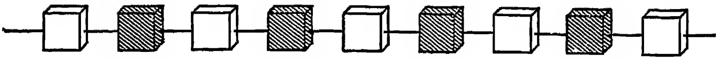
Let large plain beads indicate concepts, and small black ones between represent periods of doubt. The factor of alternation is the chief item of interest at the moment. There will be cases in which the periods of certainty are short and the periods of uncertainty relatively long:



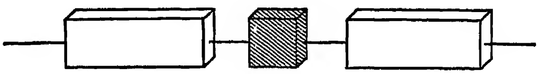
This is characteristic of youthful thinking, wherein our ideas are often being upset and displaced. There are long periods of disturbance when we do not know what we think. More characteristic of the later stages in the maturing process would be a situation just the reverse, long static periods broken by uncertainties which are short in duration because the mind is better trained and knows how to face difficulties:



During that period when we are encountering new ideas rapidly, when the great intellectual efforts of men are first made accessible, the alternations will be frequent:



This would describe our college days. But before and after this period of great excitement, while we are still protected by our home environment and after we are more or less settled in our convictions and are ourselves providing a home environment for our children, the alternations are few:



But whenever thought matures, slowly or quickly, or with greater or less certainty, this structure of alternation will be observed. It is the groundwork on which to base our pattern of dynamic thought. We are in a position to state the first principle of this new logic. *The structure of maturing thought is an alternation between periods of certainty and periods of doubt.*

We are like people driving through the heavy traffic of a modern city. We are stop-and-go thinkers. Stop-and-go driving consumes gasoline rapidly and, similarly, dynamic thinking spends much mental energy. But both are necessary expenditures. The doubts which we encounter are the red lights which warn us of difficulties ahead, and sometimes there are long impatient waits before we get the green signal again. But if the signal for doubting and suspended judgment were never flashed, if men moved continuously ahead in the false serenity of childish ideas, we should be in as much of a mess as if all traffic lights forever flashed green.

The Second Principle: Stepping-Stones Across a River

Little Johnny Smith is brought up to believe that certain actions are right and others wrong. Perhaps he has gone regu-

larly to Sunday school and been taught this. Perhaps his parents have used moral persuasion, or the woodshed. His problem is clear and easy. He has only to do what he is told.

Then one day he wakes up to the realization that he has a conscience of his own and that it does not always agree with the commandments he has been told to respect. Is it sinful to play cards on Sunday? He has been told so, but when he starts experimenting with the idea he is surprised to discover that his conscience does not trouble him. He resolves to obey his conscience. But after a while he begins to wonder about this conscience business. Is conscience infallible? He is alarmed to find that the dictates of his conscience are different from those of some of his school friends. And a little more experimenting shows that by experience and effort he can produce alterations in his own conscience. His conscience used to bother him about things it winks at now. He is in another period of doubt. How will he know what is good and what is bad? By consequences? He works out a theory in terms of the greatest good for the greatest number. But then he begins to worry about intentions. Is it intention or consequence that makes it wrong to steal? Why not steal a little from a rich man who would not feel the loss? No bad consequences here! Then either he hears about Kant or he works out his own version of the categorical imperative. Certain actions are good for their own sake, and their goodness has nothing to do with consequences. He who seeks rewards for being "good" is really far from righteous. Then practical doubts may arise again. We cannot follow Smith indefinitely. He may end up with Kant or with Spinoza or obedience to an orthodox church. But he gives us a concrete example with which to work.

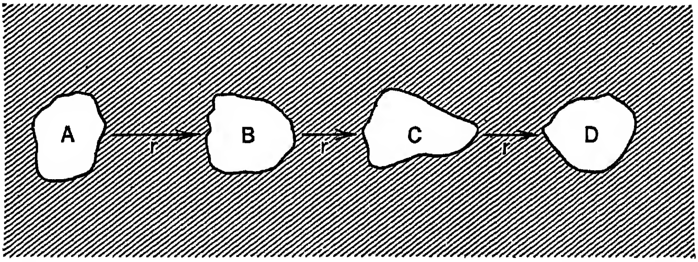
One of the first things to observe about the intellectual history of Johnny Smith is that it is not just an alternation of certainty and doubt. The certainty is a positive thing and the

doubt a negative thing. The structure would lose meaning if it were symbolized just as an alternation:

C–D–C–D–C–D–C–D–C–D–C–D–C

The whole point in Smith's history is that he changes from one certainty to another: commandments, then conscience, then consequences, then categorical imperative, etc. It is as if Johnny were crossing a river on stepping-stones. He leaps from one stone to another. Each stone is different and must be treated differently if Johnny is to make progress; but the leaps between stones are all the same, just leaps.

And so it is in Johnny's mind. His certainties are all different and must be given different symbols in the expression of the structure; while his doubts are all the same, just doubts. The state of doubting is the same whether you doubt commandments, or conscience, or consequences, or categorical imperatives. So now we will change from the analogy of beads on a string and consider the stepping-stones idea more closely. This suggests another and better diagram for the treatment of the problem of conduct, and a more accurate set of logical symbols:



To use words with which we are already familiar in other parts of logic, we may say that the A's and B's and C's and D's, and so on, are terms in this dynamic structure, while the r's are

relations. The structure, then, is one in which terms alternate with relations.

Perhaps we know a Sammy Jones, more hard-headed as a youth than Smith. He did not go to Sunday School because he lived on the East Side of New York and his first principles of conduct were picked up from experience. He may have got a cuff from his mother or a kick from his father, but only when he was in the way. Probably his first idea was that anything was O.K. as long as he could get away with it—consequences. Later he may have found his way into the local Y.M.C.A. and been told something about conscience. And if he was a bright boy and got the breaks, he may have gone to high school and learned about other possibilities. At any rate, his intellectual history is far different from that of Johnny in its content. *But not in its structure.* Here also terms alternate with relations. The important point to note is that while the terms have changed places, the relations of doubting are the same. The A's and B's and so on may stand for any ideas; principles of conduct, concepts of God, evaluations of music. But the relations between them will not vary. If maturing thought has any structure at all it will be found in the uniformity of the r's. And we may now state a second principle of this new logic: *The structural relation between ideas in maturing thought is always the same.*

In one respect at least thinking is like eating. Most people in an average community eat three meals a day at approximately the same time. But they eat different foods. The time relation between the meals is much the same, but the contents are quite diverse. Of course the individuals in one family eat the same fare, if mother is not too tenderhearted. But this only strengthens our analogy. People in the same family are apt to think the same thoughts, too. Johnny Smith and Sammy Jones are different individuals, and each of their thought processes will

show in its content the individual stamp. It is only the relation between various elements of content that is the same. This relation remains the same throughout the series of any one individual and from individual to individual.

The Third Principle: The Logic of Contradiction

It is one thing to say that there is a structural relation between ideas in maturing thought and another to find out what that relation might be. We know that it is a relation which involves doubting, but this does little more than label it so that we recognize it when we see it. We know from other parts of our logic that a relation is characterized by the properties of the terms which it brings together. For example, the relation of symmetry brings together certain types of terms, and emphasizes a connection among them. We cannot take *any two terms* and relate them symmetrically. The two halves of the letter "A" are symmetrical: two halves of the letter "L" are not. If we examine a number of the terms connected by the same relation we are usually able to characterize that relation.

Let us then, examine ideas as they participate in a maturing series. There is obviously a definite connection among them, because we cannot meaningfully so relate *any* group of ideas. There could not, for example, be said to be a process of maturation moving from "Goodness is defined in terms of consequences" to "The moon is made of green cheese" to "All leopards with monocles are dangerous." A maturing series of ideas will deal with the same problem, will treat of the same general subject: "Goodness is obedience to commandments," "Goodness is obedience to conscience," "Goodness is that which produces the best consequences," "Goodness involves intention," and so on. Furthermore, a series of ideas would not be said to

mature if each idea in the group simply reëchoed to others: "That is good which helps us to survive," "That is good which has the safest consequences," "That is good which leads to self-preservation." A series of maturing ideas must be the same in that all members deal with the same subject, but must have difference in that all members say something different on that subject. In other words, the ideas in a maturing series mutually exclude each other. *In maturing thought only one of the ideas in the series can be accepted as true: the truth of any one makes the others false.*

But this is not all. It would not be profitable to Johnny Smith to go through the series of concepts of the good life if he ended by believing them *all* to be false. If he is to act thoughtfully, and not just obey the whim of the moment, he will act in terms of some judgment about the good life and in doing so will be giving allegiance to some idea about it. As a general rule it may be said that the last term in the series is the one regarded as true, and that each term before it has had its day. *In maturing thought the last term in the series is, temporarily, regarded as true, and each term preceding it was at one time the last term and held to be true.*

In the structure of maturing thought, then, only one term will at the moment be regarded as true, but at least one must be so regarded. In other words, they cannot at one time all be false or all true. There is something here reminiscent of relations between pairs of propositions. When we found two that could not both be true and could not both be false we called the propositions *contradictory*. In its original meaning this relation of being "contradictory" applied only to pairs of propositions: each proposition had only one contradictory. But if we take the liberty of extending the meaning to include more than two propositions, it will be seen that we get just such a relationship between propositions as that which we have been trying

to describe. Hence we may assert a third principle of this logic. *The structural relation between the terms in a maturing series is that of contradiction.* It is for this reason that the logic of maturing thought, the moving picture of the structure of our thinking, is sometimes called the *Logic of Contradiction.*

This is a principle at once important and dramatic. All of the logic we have studied up to this point has been based on the fundamental law that contradictories shall exclude one another from any logical structure. Wherever contradictions were found the structure was invalid. If, on the basis of the same fundamental assumptions, you could prove both that monkeys have two tails and that they do not have two tails, you knew that something was wrong with the structure of your thinking. Contradiction in a system was not just a danger signal—it was a complete wreck. The logic of any cross-section of our thinking, of any static system, is called the *Logic of Non-Contradiction*, and its entire effort is in the direction of removing contradictions. But now we have made a unique discovery. In the logic of maturing thought, in the moving picture, contradiction becomes a positive element rather than a weakness of thinking! Johnny Smith, in embracing a new concept of the nature of goodness, renounces or *contradicts* an earlier judgment. Contradiction is a symptom of healthy development.

The more mature opinion is always a contradiction of what has gone before. Where there is no contradiction there is no growth. It is the old principle of not being able to eat your cake and have it too. Contradiction is destructive, but it should be encouraged, not shunned. This was the secret of Socrates' success as a teacher. But it goes against human nature. And that was the reason for his death sentence. An old teacher of mine used to say that the things we forget are more important than the things we remember. It is also true that in some respects the ideas which we discard are more important than the ones

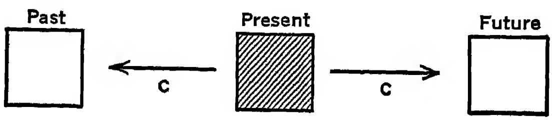
we believe. It is the judge-a-man-by-his-rubbish-pile idea carried into logic.

The Fourth Principle: Chemistry and Maturing Thought

The ideas which we discard *are* important. They are now a part of yesterday. But to-day would not be to-day if there had not been a yesterday. The old ideas are discarded, but their influence carries into the present.

Johnny and Sammy each believed at some time that the goodness of an action might be found in its consequences. But for Sammy this was the first concept of goodness that he encountered when he started thinking. For Johnny, in contrast, it came later and was seen in the light of discarded ideas of obedience to commandments or to conscience. For Johnny, then, *the same idea is a more mature one than for Sammy!* Dynamic thinking is a cumulative process. Yesterday's judgment that Tschaikowsky is a great composer is a structural part of to-day's judgment that Brahms is great. And to-day's evaluation of Brahms will, if your understanding continues to function, become a structural part of to-morrow's taste in music. Our present thinking employs past thinking as its material, and in turn becomes the material of the judgment which follows. You could not approach Brahms as you do if you had not previously been acquainted with Tschaikowsky. Freshman themes are essential to later writing.

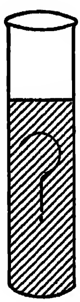
And here logic strikes human nature a telling blow. Looked at from the point of view of present thinking there is a relationship of contradiction in both directions, past and future:

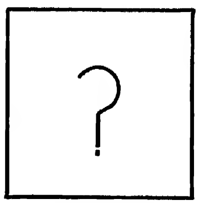


But because it is historical, because it has already occurred, the former gets more recognition than the latter. In the greater maturity of our present thinking we can look back with condescension upon what we once thought. Do you remember the type of movie you liked when you were in high school, the kind of party you enjoyed? Our relationship to yesterday is clear, because to-day is in a definite sense a product of yesterday.

But our relationship to to-morrow? That is a different story. But why stop with to-day? The logic of dynamic thought stretches in both directions, into the future as well as into the past. We must always consider to-day's thinking as the material of something quite contradictory which we will think to-morrow. It is futile to prophecy. Who will be your next favorite composer? To what new understanding of the nature of goodness will you progress? You cannot know until to-morrow comes. But it is blindness and narrowness and laziness to suppose that we ever know all that there is to know about any problem. Look to the future in your thinking!

This raises a second dramatic distinction between the new logic and the other. We have seen that the logician in studying deductive systems is always faced with the same problem. Given certain premises or postulates, what conclusions can be drawn from them? He is like a chemist making an analysis of some unfamiliar compound in a test tube:

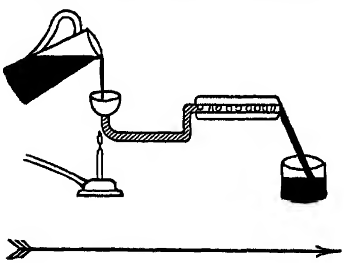


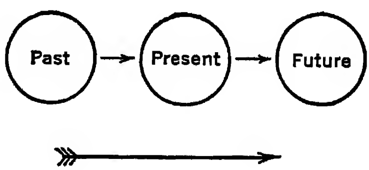


Deductive
System

The problem is entirely static. The compound to be studied is there in the test tube and the chemist's job is to analyze it, to discover what elements are there. And so the job of the logician is to discover what propositions are contained in a postulate set or a group of premises. But there are certain achievements of which analysis is incapable. For one, it is not creative. The analysis of a symphony does not create a new work of art, indeed it can be, and too often is, deadly. The analysis of a geometrical theorem carries one back to the initial postulates, reverses an earlier process of elaboration.

But the artist creates something novel. Progress in thinking is progress from one postulate set to a new one. Man does not cease to be an atheist simply by analyzing atheism. A good sound analysis is invaluable in showing the inadequacies of a given position and it, more than anything else, promotes doubting, as Socrates knew. But synthesis is necessary if one is to move forward. No amount of analysis *by itself* will tell Johnny Smith the nature of goodness, though it may tell him that now is the time to move onward. Analysis tells you what you have got, or better, what you have not got; but it does not look to the future. That is why the logic of deductive systems, the logic of Non-Contradiction, is so sterile. And that is the great need for the logic of dynamic thought, the logic of Contradiction. This new logic is synthetic. To find a parallel in chemistry, it is like a process in which something is always being added and something always being thrown off:





The logic of deductive systems is sometimes called Analytic Logic, and the logic of maturing thought is generally referred to as Synthetic Logic.

Unfortunately we know little more about this synthetic process than that it is synthetic. Knowledge within a fixed deductive system, such as euclidean geometry, is always self-contained. If you want to know the properties of the interior angles of a triangle you analyze the concept, "triangle": the answer is there if you can find it. But once you have found conscience inadequate as a guide to the good life, no analysis, no matter how exhaustive, will tell you what the next step should be. For this reason we know that the dynamic relation in thought cannot be analytic.

However, the new concept does not come entirely out of the blue. It is a synthesis of that which is discarded, *as something discarded*, and that which is novel, *as something new*. The doctrine that goodness is determined by consequences, coming late in John Smith's series, is because of his past, a relatively mature judgment. The same doctrine, coming quite early in Sam Jones' thinking, is relatively immature. Hence the fourth of our principles. *The structural relation among the terms in a maturing series is also that of synthesis.*

Analysis and synthesis are not the same process moving in opposite directions, backward and forward. Whether you take a watch apart or put it together your task is analytic. It will come apart only in one way, and the parts no matter how scattered will go together again only in one way. And this way is forever fixed by the design of the watch. But the inventor who constructs a new machine is confronted with a synthetic problem, the production of something never before in existence. Hence the logic of analysis and the logic of synthesis are different in kind, not just in direction. The former deals with something already completed which will stand still so that you may

study it: the latter is concerned with creation and novelty. One must see this difference clearly to appreciate the uniqueness of the new logic.

The Fifth and Final Principle: A Plea for Tolerance

For all of its essential and important values, the logic of analysis and non-contradiction has done almost irreparable damage by suggesting the existence of a single deductive system which shall possess final Truth. Men have fought, killed, and died for ideas because they did not understand that deduction is a circular argument, that analysis does no more than leave you with a more detailed understanding of what you already knew.

Knowledge is relative and tentative. The one most reliable sign of ignorance is the closed mind. No man, no matter how wise, can achieve a doctrine which does not reach out toward the future and greater wisdom. Every good teacher is an eternal student and knows better than to lay down the law or indoctrinate. Every judgment has a time dimension and should be regarded as a pause in an endless journey. Truth is a haven whose comforts fade forever and forever as we move. The one most reliable sign that thinking is infinite is the history of science or of philosophy.

A generation ago scientists asserted confidently that they knew all of the major facts about the physical universe. All that was left was to fill in minor details. Then came electrons, protons, positrons, and neutrons, and with them vistas of which scientists had hitherto not even dreamed. Scientists no longer boast that they have found the "ultimate unit." They know that the quest of science is endless.

Intelligent men have for centuries been thinking about the

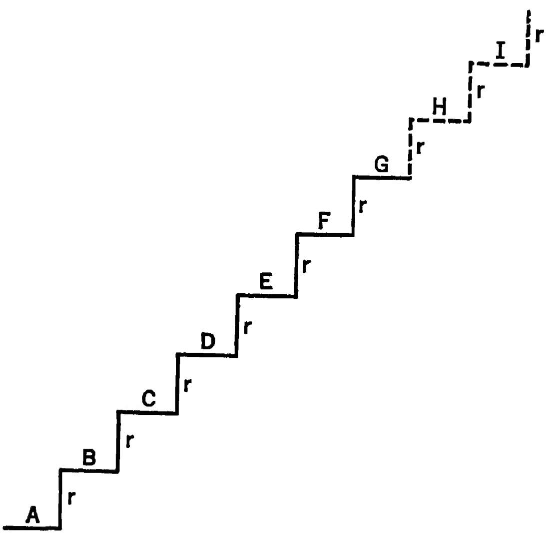
problems of philosophy, yet are no nearer Truth than in Greek times. They are more mature in their thinking, but that is something else. The best philosophers disagree and will always disagree. There are no final answers. The realization of this has discomforted philosophers (and their students!). But they have only themselves to blame. For centuries they have taught a logic which has led men to expect finality in knowledge. They have taught that all we have to do is to analyze our judgments and avoid contradictions, and we will discover the Great Truths. Until recently all philosophers, and every one else for that matter, had proceeded in his thinking on the principle that finding Truth was like taking money out of a safe. It was there to be taken: all we had to do was to learn the combination. Every philosopher had commenced with certain assumptions, analyzed them with care and organized them into a deductive system, then sought the Truth at the basis of the system. This is just like putting money in a vault, slamming the door, then working feverishly to open the door to see what is inside! You should not be surprised by what you find.

There is nothing in the structure of maturing thought which suggests that the process has an end. This is in many respects the most important of all of the characteristics of the structure of dynamic thinking. It recommends, even more than earlier considerations, tolerance of the thinking of others. There is no point at which we must not continue to doubt and question, to accept the possibility that our opponent has gone further in his thinking than we have in our own. We must never be too sure of ourselves, for we shall never know all of the answers. We laugh at the ignorance of our great-grandparents; but it would be more healthy to join prematurely in the laughs that our great-grandchildren are going to have at our expense. Hence the fifth, and most important, of the principles in this new logic: *The process of maturing thought is an infinite process.*

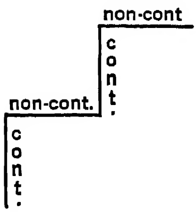
To reduce this to a rule of thumb: *There are at least two sides to every question.* Too bad Martin Luther could not have realized this. He looks small to-day against the stature of Erasmus of Rotterdam.

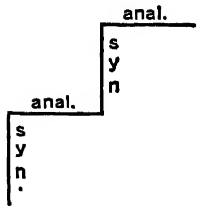
A New Diagram of the Structure We Are Studying

We are now in a position to construct a diagram of maturing thought more illuminating than any we have yet attempted. To mature in one's thinking is like taking steps upward on a flight of stairs. The terms in the series are represented by the horizontal treads of the stairs and the relations are shown as sudden rises to new and more mature levels:



A detail of such a diagram will show how non-contradiction and contradiction, and analysis and synthesis, work together:





On the plateaus we employ the traditional deductive logic, elaborating by way of analysis an already acquired set of judgments and avoiding contradictions. These stages are static, and, for most of us, complacent. But they are essential to profound thinking. The more we analyze our position the more we understand it. And the more we understand it the sooner we see the inadequacies and begin to doubt. And when we begin to doubt we are ready for another step upward. In the early part of the series the horizontal stages will be shorter, for in youthful thinking one's position is not well thought out and inadequacies are more easily detected. As our positions become more mature they undergo more and more successfully the test of analysis.

The dynamic, and more humble, stage is that in which we courageously leave one plateau and work toward another. This is the process that all education in general, and all logic in particular, should emphasize. In this stage contradictions are welcome and new syntheses sought. It is during this period that significant and exciting intellectual events occur. The moments in our mental histories which we remember are those crucial ones in which we suddenly realize that we must bid old acquaintances good-by and set off again on the road. Intellectually we should be travelers, not stay-at-homes; and some of our longest journeys should be taken while we are young.

If a student asks me why I believe in the reality of man's spirit, no answer which I yodel from my plateau will satisfy him. But if I put myself in his position and remember what I

was thinking at his stage and the contradictions which I encountered, in other words, if I discuss the dynamic syntheses which have been involved in the working out of my present position, then he will see a dynamic relation between his breathing place and the possibilities ahead. Education works through the challenge of possible ascents, not with the passive acceptance of a teacher's more mature philosophy. The student must be shown how to climb, not hauled up on a rope. The logic of maturing thought is the nucleus of a philosophy of education, and one which does not take it into account will be sterile and dogmatic.

Hegel and the Logic of Contradiction

Suppose all histories of philosophy were destroyed in a world conflagration. Suppose we knew nothing about man's past philosophic efforts. Could we reconstruct that part of our history? There are some who believe that we could, that just by studying the structure of man's thinking we could infer what its content must have been. Perhaps this logical archeology is no more fantastic than reconstructing a primitive culture from a few fragments of pottery.

Among the first to do serious work on the logic of contradiction was one who believed firmly that he could write the history of philosophy from a knowledge of only its dynamic structure. His name is Hegel. His outline and time-table of what the maturing mind should think is too complex to discuss here in detail. It was roughly an art-religion-science-philosophy succession. He argued that all logical thinking follows the same sequence in its development. Perhaps he was carried away by his enthusiasm for the newly-discovered structure. It is as if we were to say that every Johnny Smith and Sammy Jones must go through the same series of concepts about the good life, and in the same order. In that case the measure of a man's

maturity would be apparent immediately. "Oh, he is still in the stage of thinking that goodness is measured by consequences. I used to think so too—when I was a sophomore."

This manner of thinking is so tempting—and so false! Most of the freshmen whom I encounter believe belligerently in God: later they become atheists: and still later they come back to an altered concept of God. But we all know individuals who have started out as rabid atheists, found Religion, and later decided that God has no meaning for them. The content of any individual's series depends in part on his early background. If the latter sequence is less frequently met in college than the former it is because the orthodox religious point of view is more prevalent in the homes from which my students come. The fact that most undergraduates are Republicans has the same accidental cause. It has nothing to do with a logical principle.

The content of any individual series also depends in part on what that individual encounters. You may meet Karl Marx long before you know Adam Smith, but it is usually the other way around. Some get into science laboratories early, and the experience leaves a lasting impression: others come to understand the scientific attitude much later. When did you meet St. Francis? When Rousseau? If, as is sometimes the case, a student gets out of college without knowing the difference between Marx and Lamarck we may rightly be worried about him. I should be worried if he had not read Tolstoi and Spengler. It is the job of the college to acquaint the student with the major intellectual forces of his time. But acquaintance is not always understanding. We forget that often the most vital stages in the maturing process take place when the student gets out into the world. Sometimes the accidental reading of a book will start a whole train of thinking, open up a new world. Thoreau's *Walden* has done this for many. An odd friendship, a tragic experience with death, a trip to Europe: these have

similar results. So many of these things are chance happenings, and they are all so different in their effects, that one could hardly expect two individuals to develop the same set of ideas in the same order. The logic of the process and its content are quite separate considerations.

Not only is the order of ideas in the maturing series different from individual to individual, but final outcomes differ widely from one to another. Hegel did not acknowledge this, either. If one thought maturely, then one thought as Hegel did. This attitude is tempting. When my colleague in a science department tells me that philosophy represents the raving of the adolescent mind I set my mouth in a grim smile of pity, but that grimace is entirely illogical. My moralist friend tries to be helpful by pointing out that all philosophy leads inevitably to religion. Perhaps I could counter by giving him a small idea of the thoughts that go through my mind when the minister starts ordering God around in his Sunday morning prayer. But where is the help in that? The point is that all three of us have experienced sequential progress, and human nature (this thing that logic is always fighting!) being what it is, each of us is going to be confident that he is on the right track. To each the positions of the others seem immature. But in our rarer and more logical moments we pay respect to whatever a man thinks *so long as his thinking is mature.*

How Measure Maturity?

So long as it is mature, that is the point. But maturity is determined by structure, not content. There are many who vote the same ticket you vote at an election without being mature about it. Ask any politician. Why, then, is it not reasonable to suppose that after mature thinking some vote the other ticket? I know of no measure of maturity that does not inevitably

refer back to the diagram of the flight of stairs, and ask only that a man be willing to doubt, to cast old ideas by the wayside, to move from one plateau to another, to know what he thinks and why. The test is "Are you moving up some staircase?", not "What stairs are you ascending?"

It is not what a man thinks, that counts, it is how he thinks it. One's own vital and personal acquaintance with the process of maturation that one has undergone invites the weakness of believing that mature persons will think alike. Also, one recognizes in the judgments of others positions which one once held. If you at one time wore a hairshirt and preached rugged individualism but are now socialistically inclined, you will feel superior to the hair-shirted individualism of some one else. You "saw through" that foolishness. But your friend may have a far more profound understanding of individualism than you had. He may be a mature individualist.

The thinker is independent; he thinks for himself. He is the active formulator of his ideas, not the passive reflector of the ideas of his teachers, his friends, or the books he reads. He need not be the originator of his ideas, any more than he need be the cultivator of the vegetables he eats. Our intellectual heritage is so rich that originators, like gentleman farmers, are few. But he knows what he thinks and why. He knows the value of doubting. He knows the importance of abeyance of belief. He is venturesome: he welcomes mature opinions divergent from his own and gives attention to them in proportion to their divergence. His thoughts do not simply accumulate, like drops of water in a rain barrel. They change, with a pattern that can be symbolized in a diagram. *They change.* That is the important point.

There are, of course, general trends. (1) *The superficial will come before the profound.* It would probably not be illogical to dismiss as immature a lover of the efforts of Edgar Guest.

The economic pronouncements of the local Rotary Club, or the foreign policy of the American Legion, may often be similarly treated. But superficiality may be characteristic of any doctrine. There are superficial capitalists and superficial communists, and profound capitalists and communists. Rousseau was never profound, but the ideas for which he stood are often penetrating. However there are some things, like the average moving picture or Snappy Stories, which represent sheer superficialities. (2) *The simple will come before the complex.* Rousseau will surely have an earlier and less mature effect than Kant. In music the brass band is easier to listen to than the quartet. One would not start right in enjoying Bartok. One has to acquire a technique of listening. The art of thinking also develops slowly. It is to be expected that one will think simple thoughts before difficult ones. (3) *The emotional will come before the intellectual.* There is, of course, a biological reason for this. We feel deeply before we think deeply. Most of us go through early romantic periods. There is also a social reason. Emotionalized attitudes are widespread in the community. Most people are dominated by their emotions. But emotion, like superficiality, may attach itself to any point of view. One may be just as emotional in one's atheism as in worship of a deity. One may be just as emotional about Bach as about Sousa.

Advances in Science and in Logic Compared

It has been suggested that we made great advances in the field of the exact sciences long before we even began to work seriously at the biological ones because the two kinds of science involve quite different types of concepts, and because those of the exact sciences (the quantitative ones) are more easily manipulated. Men like to count and measure. Perhaps there is a parallel to this in the field of logic. The tentative principles we

have set down do not go far, but it is hoped that they indicate a direction which the study of the structure of thinking may profitably take.

Of the characteristics of mature thinking, that of synthesis is at once the most far-reaching and the most bothersome. It is responsible for a lack of definiteness which may be unavoidable. Surely analysis is neater, easier to handle. Perhaps we understand analysis, and hence the logic of non-contradiction, better because we have studied it more carefully. And perhaps we have studied it more carefully because it is easier to manipulate. Perhaps, on the other hand, synthesis is by its nature more difficult to describe and less definite. It has always been considered more arbitrary and subjective. This, I think, is untrue. The logic of maturing thought is just as logical as the logic of deductive systems. But we may never be able to say as much about it. We do not know yet: we have not studied it enough. But we do know that this new logic is vital to an intelligent understanding of what goes on when we think.

Chapter Fifteen

LOGIC AND THE HISTORY OF PHILOSOPHY

IF ANY ONE came to you and asked you to describe a spiral staircase, you would make a few familiar gestures in the air with your forefinger, then give up and take your friend out to see such a phenomenon. Some things are more easily studied by way of examples than through descriptions. I suspect that the logic of maturing thought is one of these things. This is because it is a process, a movement. If I knew you personally I might draw exciting examples from your own thinking. It is unfortunate that I cannot, because nothing would be more profitable. Although I know my own thinking quite well and could give what seem to me to be excellent examples from it, I shall not do that either. Teachers talk too much about themselves. So I shall turn to the history of philosophy for illustrative material.

In doing so I suspect that I shall not be doing badly. In your process of mental growing up, you have gone through something very similar to what men have experienced historically in achieving a relatively mature way of thinking about philosophic problems. The history of philosophy illustrates strikingly and on a large scale what is going on in the mind of the maturing individual. If you paid attention to what I said in the preceding chapter you will realize that in making this assertion I am referring to the structure and not the content of thinking.

The first of the philosophers, Thales, said that everything is made of water. I doubt if many of you thought that, even in extreme youth. But in your school days you were as naïve as Thales. No need to feel ashamed. We have all faced perplexities with wild speculations. Can you remember wondering childishly what made an automobile move, and deciding that it was the impact of the exhaust against the ground? I am acquainted with one who can. He knows better now. As you mature I shall not expect you to write an *Ethics* or a *Critique of Pure Reason*. It is in the structure rather than the content of your growth that you will parallel the history of philosophy.

We gradually acquire wisdom, but the process is slow. A college freshman is as immature as the pleasure-seeking Aristippus or the superstitious Pythagoras. He needs a Socrates. In fact, if he or his parents know what is good for him, that is why he came to college. The sophomore has had a lot of new experiences and, as materialist or Platonist, thinks he knows all of the answers. Or he may be a skeptic, and assert that there are none. The senior, we hope, gains a perspective that is relatively mature.

History and the Logician

But the history of philosophy is dynamite to the logician. We are all lovers of explosive exaggerations. They give color to the world. The connection between the logic of maturing thought and the history of philosophy is so intimate that I am tempted to suggest that logic be studied in courses in the history of philosophy and that the history of philosophy be taught in logic classes. It could be done. And, since I have made the suggestion and am writing a book in the field of logic, I am more or less challenging myself to show that there is something in the idea.

I do so at great peril, however. There are more chances of going wrong in interpreting the history of philosophy than there are ways of skinning a cat. It is so easy to make the mistake of forcing the individual to recapitulate in the content of his own thinking the content of the history of man's thinking. And it is even easier to perceive in the history of philosophy a structure which is the "key" to all philosophic problems.

But if we will keep a stern eye on ourselves, and stop short when we are in danger of exploding our dynamite, we should be able to play with it with considerable profit. From the many periods and manifold problems in philosophy's history, we shall select an episode which illustrates particularly well the maturing process. The period is early Greek. In early periods events happen rapidly and are easily studied for their dynamic features. The problem is familiar: What is the nature of nature? A familiar problem, one which figures in all of our thinking, should have points of contact with individual perplexities.

From Thales to Anaximander to Anaximenes

The important thing in studying the history of philosophy is to see the thinkers of the past as they saw themselves. Thales believed that everything is made of water. Sounds silly, does it not? But it was not silly to Thales. Water was the only familiar material that changed readily from liquid to gaseous or to solid state. Water is necessary for the growth of plants. Man depends for his life on being able to get water. Some have pointed out that Thales lived on an island, and must have been impressed with the vastness of the amount of water in the world. And water is everywhere: in the sky, on the earth, and under the earth. It was a good first guess.

So much for a beginning. But now see what happens in this study of nature. See how the various solutions follow on each

other, and why. The development is one of the most fascinating recorded.

Thales' guess was so naïve that it was soon doubted. Already we are getting an alternation between certainty and doubt. How could water be anything but water? When it is fire it has the qualities of fire and no longer those of water. Anaximander saw the force of this argument and realized that if water were the substance of fire and earth and air, it would have to change its properties. How, then, could we still speak of it as water? What we need, he thought, as a solution of this problem is a substance which will have *all* of the properties found in nature, one that will be boundless in its possibilities. Hence he called the underlying substance The Boundless. It was boundless in space and time also, but this is another argument. Again a period of certainty.

And once more a period of doubt. This time it was Anaximenes who asked the questions and began to wonder. Was the answer which Anaximander gave really any answer at all? Did it not raise the same difficulties that Thales had faced? Thales had started with a boundless number of properties and tried to explain them by one underlying substance. When this failed, did not Anaximander simply go back to the boundless number of properties and give them a name but no explanation? Anaximenes thought so, anyway, and he returned to Thales' fundamental idea that there must be a single underlying substance. But Thales had offered no explanation of how there could be a single underlying substance which would be responsible for the boundless properties found in experience. Could such a principle be found?

Anaximenes hit upon the idea of condensation and rarefaction. If there is a single substance it can only differ by there being more or less of it. Now there is one familiar material that is easily condensed and easily rarefied. Air. Air is also necessary

to life: breathing is perhaps the most vital of our functions. In fact, air is more necessary than water. We can live longer without water than without air. A little ingenuity will bring forward other arguments in defense of air as the underlying substance. It is intermediate between the heat of fire and the coldness of earth and water. Air seems warm when rarefied: blow lightly against your hand and you get a warm sensation. And it seems cold when condensed: blow hard against your hand and you feel cold.

Four of Our Principles Already Illustrated

We may profitably pause at this point and consider the logic of the argument outlined. We have already remarked the alternation of certainty and doubt: Principle One of the structure of maturing thought. It is also evident that the periods of doubt are, as such, exactly alike; that they are the structural ties which hold together the three contributions thus far made to the problem of the nature of nature. Principle Two.

Notice next that these three doctrines are contradictory in a broad sense. Each excludes the possibility of the other two. The underlying substance is water *or* it is the Boundless *or* it is air. Pay your money and take your choice. If you had lived in Pre-Socratic times, you might have agreed with Thales or with Anaximander or with Anaximenes, but with only one at a time. We are strictly monogamous in our marriages to philosophic doctrines. Principle Three.

Notice also that there is a synthetic relation among these three stages of the problem. The full meaning of what Anaximander said cannot be understood until you know the thinking that preceded it. Thus, while Anaximander contradicts Thales, his thinking also contains as an integral part an understanding of what Thales did. It does not just rise up suddenly and in-

explicably, like a jack-in-the-box. It is dependent on Thales' contribution. And, in the same manner, Anaximenes is synthetically related to both Thales and Anaximander. The process is clearly cumulative. Anaximenes could not have thought as he did if Anaximander had not thought as he did; and Anaximander could not have made his contribution if it had not been for Thales. Furthermore, while an analysis of Thales' solution was essential to what Anaximander did, the answer to the difficulty did not come by way of analysis alone. It involved a new idea, was a synthesis. Principle Four.

The last of our Principles, the fifth, does not appear until we reach the end of our example. But watch in what follows the recurrence of illustrations of the other four.

Heraclitus and Parmenides

All who have studied Heraclitus and Parmenides know that the point to which they carried the metaphysical problem was to set a style for the entire history of philosophy that followed. We can concern ourselves with but a small part of their contribution.

Heraclitus, like Anaximander, was impressed with the importance of change. Everything in nature is changing. There is nothing in our environment that will not be different in a few years. Even the "eternal" hills are washed slowly into the valleys. As he put it in a famous metaphor, we never step twice into the same river. And what could be a better symbol of change than Fire? Fuel is always being drawn into the flame, altered by combustion, and given off as waste in another form. The universe is a closed circuit in which an ascending and a descending current counter-balance each other: there is an opposition of motions. Change is rhythmical and kept within bounds by law. But change, that which is observed by the

senses, is the significant thing. There is no unchanging substratum.

Parmenides swung back to the side of Thales and Anaximenes.. He was able to state the problem more maturely, however. What can we mean by "change" unless we assume that there is something continuous and unchanging in the process? If you see an automobile at the top of a hill, look away, and then a few minutes later see an automobile at the bottom of the hill, you do not think that change has taken place, that the automobile has coasted down the incline, *unless* there is sameness in the situation, unless the two automobiles are of the same make and model and color. And the same thing is true of combustion. We do not say that wood has "changed" into ashes and smoke unless we mean that there is something underlying which is unchanging. We should not have change if first we had wood, and then ashes and smoke, but the two were substantially unconnected. This is a more sophisticated way of putting the matter than Thales or Anaximenes had conceived. Parmenides has made one contribution to a maturing idea.

Parmenides avoided Thales' and Anaximenes' difficulty more adroitly than had Anaximander. He realized that water and air could not be satisfactory solutions: water is water: air is air. Something must be found which can take on any or all of these various properties. It did not pay to be to specific. But instead of referring to The Boundless, Parmenides chose the opposite concept. He spoke of substance as The One, or Being. The implications of this are great. The change which we observe with the senses becomes interpreted as illusion, and that which is real in nature can only be known by reason. We cannot see the One, but careful intellectual study shows that it must underlie change if change is to be understood.

The Maturity of Parmenides and Heraclitus in Comparison with Their Predecessors

Parmenides had faith in the power of reason: Heraclitus had faith in the senses. Parmenides believed that reality is permanent and unchanging: Heraclitus believed that reality is in flux. Parmenides believed that reality is one stuff: Heraclitus believed that it is composed of the Many. Men were now faced with two philosophic problems of major importance. One involves the distinction between appearance and reality; the other, the distinction between the senses and reason as instruments of knowledge. In other words, the problem with which we started has now reached more mature statement, unfolded more of its implications. The conflict between these two philosophers had brought the realization that a philosophy of nature must account for both change and sameness. If there is an underlying substance, it certainly changes its appearance. If there is such a thing as change, there must also be something permanent. The metaphysical problem has become appreciably more mature, but there is still room for advance in this direction.

From Empedocles to Anaxagoras to Democritus

We may not wilfully wave change out of existence. Change doubtless exists and has to be explained. Reality must somehow be many, or we could not observe motion and change. But what about permanence? It must be retained too. Somehow a way must be found of combining permanence and change. At this point there occurred a flash of genius which has affected all subsequent thinking. It illustrates beautifully the fact that maturing thought is synthetic.

Empedocles brought the concepts of permanence and change together in the idea of *elements*. Reality is composed of elements. There are four kinds of element, Earth, Air, Fire, and Water. Permanence belongs to the elements themselves: change belongs to their shifting relations. The appearance of change is produced by alterations in combinations of units which do not themselves undergo change. Appearance is accounted for: change is accounted for. The senses do not mislead us: the demands of the rational understanding are satisfied. This development of the concept of underlying elements is one of the most dramatic advances in the history of philosophy.

But the problem was by no means solved. On a new level we have just a beginning, an idea as new and untested as Thales' initial idea was. How can any combinations of Earth, Air, Fire and Water produce such things as wood and iron and flesh and bone? These four elements are what they are: they have certain specific qualities. How can their few qualities account for the many thousand diversities we find in our environment?

Our next figure, Anaxagoras, saw the importance of the idea of elements but he also saw the limited application of the idea as made by Empedocles. Empedocles had made the same mistake that Thales had made, that of being too specific. Why limit the elements to four? Anaxagoras makes the same objection to Empedocles' doctrine that Anaximander had made to that of Thales. The parallel here is truly extraordinary, and its implications for an understanding of maturing thought great. And Anaxagoras makes the same correction on a new level that Anaximander had made on an earlier and less mature one. Since there are infinitely many qualities in the environment, the elements must be invested with all of these qualities. Hence the elements must be infinite in number. In any individual object the qualities will be determined by the proportions in which the elements are mixed. Water, for ex-

ample, is a mixture of elements in which cold, wet and fluid ones predominate. We may say, in short, that each of Anaxagoras' elements is a small piece of Anaximander's Boundless. He simply applied the idea of The Boundless to the new concept of elements. Again the synthetic character of maturing thought is illustrated.

Was there an Anaximenes on this new level? It was not long before men realized that these qualitative elements, or atoms, presented difficulties too. Was Anaxagoras' idea any more a solution of the new problem than The Boundless had been of the old one? By the addition of the concept of atoms it represented an advance, but the new problem offered new difficulties. Anaxagoras could explain change far better than Anaximander, but what about these qualitative atoms? Think how many different kinds there would have to be in order to account for all of the properties in the environment. Think of the variety of colors and tastes and smells and textures and temperatures that would have to be represented. Instead of explaining the variety of nature and its changes by something more simple, Anaxagoras merely put that variety and change into small particles of matter and left the fundamental problem unsolved.

So the Anaximenes of atomism, Democritus, conceived of a new type of atom, a quantitative atom. In doing so he borrowed, indirectly, from another predecessor, Parmenides. These quantitative atoms were made of one substance which was permanent, but which had none of the properties that we usually associate with the environment. Democritus' atom was colorless, tasteless, odorless, without temperature, etc. It possessed only what have come to be called *primary* qualities; shape, size, mass, and velocity. And each atom differed from the others in these respects alone. The *secondary* qualities of color, taste, odor, temperature, and so on, were produced by differences in

primary qualities. This is very close to what we think to-day. We explain color quantitatively in terms of differences in the wave length of light waves: we explain sound as vibrations of air waves: we explain temperature in terms of rate of vibration.

Principle Five at Last

Democritus' concept was so powerful that it dominated scientific thinking until the beginning of the twentieth century, quite an extended plateau in the maturing of that aspect of our thinking! In this century the problem has opened up again. The atom has been smashed. Physicists are talking in terms of electrons, positrons, and neutrons. At the moment we are in a period of uncertainty in which men are unable to decide between corpuscular and wave theories of matter. The two obviously contradict: one is used in one situation, the other in another. What the next jump will be, what new concept will help us to move forward by a fresh synthesis, it is impossible yet to say. The last of our five principles has made its appearance.

But the development from Thales through Democritus will stand by itself as an illustration of maturing thought. One of the advantages in studying Greek philosophy is that we can easily see it in perspective. Other illustrations might have been used. The development of the idea of the Good Life, beginning with Aristippus; the realist-nominalist-conceptualist argument in medieval philosophy; the concept of the State from Hobbes to Hegel; the development of the empiricist's argument in Locke, Berkeley, and Hume; the rise of Utilitarianism from Hume to John Stuart Mill and thence to the pragmatists; the idealistic movement from Kant to Gentile. However, we are not attempting to write a history of philosophy. For the purposes of our present study the episode we have chosen will be sufficient.

The Historian Has a Little Surprise Package for the Logician

You must certainly have noticed that in following the development in Greek philosophy of the concept of nature we have come upon a rather unexpected pattern of repetition, one of which there is no mention among the five major principles of dynamic thought. What is the significance of this phenomenon? Have we neglected something up to this point? Not only do we have repetition, we have a kind of double repetition; repetitions within repetitions. Grab something to keep from getting dizzy and let us try to investigate this new complication.

The first type of repetition is fairly apparent. If you will give attention to the content of the thinking we have been tracing you will observe that it swings back and forth in a way which reminds one of a pendulum. All of these early thinkers can be divided into two groups: Those who emphasize Oneness and those who emphasize Manyness. Even Democritus, while he worked both into his concept, owes ultimate allegiance to the side of Oneness because his unchanging atom of pure Being is considered more fundamental than the appearances for which its actions are responsible. This characteristic of doubling back on itself occurs often in the history of philosophy. And it has led many students of the structure of thinking to draw diagrams which look like zig-zags or spirals. Men move back and forth in their thinking between two fundamentally different types of doctrine, advancing in the maturity of their thinking with each swing.

The wise sentry in *Iolanthe* sings a song which goes like this:

> . . . Every boy and every gal
> That's born into the world alive,
> Is either a little liberal
> Or else a little conservative.

It is eminently true of philosophers, as William James pointed out, that they can be divided into the "tender-minded" and the "tough-minded," the "head-in-air" ones and the "feet-on-ground" ones, the Platonists and the Aristotelians. To-day there are the idealists and the realists, the mechanists and the vitalists, those who believe that beauty is subjective and those who believe that beauty is objective. Everywhere you go in the history of philosophy you will find dualism and conflict. Spinoza has his Locke, Arnold his Huxley, Royce his James, the realists their nominalists, and so on. Not only are there these dualisms, but they seem to fit into one large dualistic pattern which takes in the entire history of philosophy.

The second type of repetition is more surprising because more extraordinary. The repetition in Empedocles, Anaxagoras and Democratius of what took place with Thales, Anaximander and Anaximenes is not sheer accident. Somehow as men reach higher and higher plateaus in their understanding they continue to solve their problems in the same general manner. The difference is the difference in the level of their understanding of the problem: the sameness is the sameness of the essential character of the problem. It is something like hunting game. The difference between throwing a stone, releasing an arrow, and shooting a gun represents a difference in level of the art of hunting. But whether your projectile be a stone or an arrow or a bullet there are fundamental problems of aim and trajectory which do not change.

This same type of repetition may be found in the philosophers who sought to define the Good Life. Ethical philosophers are roughly divisible into those who emphasize the individual pleasures of the senses, and those who emphasize the universal pleasures of the life of reason. The Cyrenaics valued the former to the exclusion of the later: the Cynics the latter to the exclusion of the former. Plato and Aristotle raised the ethical prob-

lem to a new level, performing the same function in this field that Parmenides and Heraclitus did in the other. Plato laid exclusive emphasis on the life of reason. Aristotle had more to say about individual pleasures. Then comes a later period in which the development of Stoicism and Epicureanism is a clear repetition on a more mature level of exactly the problem opened by the Cynics and the Cyrenaics. Is there some logical principle behind these phenomena of repetition?

The Cycle Pattern in the History of Philosophy

If we had time to survey the entire history of philosophy we should find still a third suggestion of repetition. Some have made an interesting case out of the argument that the philosophy of any civilization goes through a definite cycle. We know intimately Greek and modern European philosophic developments. They are strikingly similar. In both cases philosophy had to break away from religious superstition. In both cases there was an early interest in nature, Roger Bacon and Galileo corresponding on their level to men like Thales and Anaximander. The realist-nominalist controversy had its counterpart in the conflict between Parmenides and Heraclitus: exactly the same issues were being raised in a new form. In both civilizations there was the sudden awakening of fresh philosophic inquiry; once in the questioning of Socrates, and again in the doubting of Descartes. The connection between Spinoza and Plato is extremely close: both were impressed in the same way by the work of the mathematicians, both built important metaphysical systems which were similar in their fundamentals. You may draw an interesting similarity between the Empiricists and Aristotle or, if you look upon Aristotle as the great metaphysician of Greek philosophy, you may find his counterpart in

Hegel. The rise in the later Greek philosophers of Epicureanism and Stoicism and the end of the cycle in a full-blown skepticism; these surely have contemporary counterparts.

You may quarrel with the details, but the general trend is much the same. At first men are bound to the superstitions of their time. Then they discover that some of the idolatries are incorrect. There is an awakening of independent individual thought. Men learn to think for themselves. Once they have declared their independence, men begin to question the traditional morality. With increased maturity the scientific inquiry becomes metaphysical. The systematic philosophers arise, expressing the same fundamental issues on a still more mature level. The conflict between metaphysical systems leads to the focusing of attention on the theory of knowledge, and inevitably men learn that all metaphysical systems are based on arbitrary assumptions. Hence skepticism. But men must live and act, and the last stage exhibits practical philosophies of conduct which are essentially either sensual or ascetic.

What Happens if You Regard Thinking as Structurally Repetitious

But is this the logic or the biology of thought? Shall we find the explanation of these persistent repetitions in the structure or the content of thinking? Does the history of philosophy exhibit a logical pattern?

Any one familiar with vital statistics can tell the senior when he graduates from college how many of his class will live to be seventy years old. But will *that senior* live to be three score and ten? That is another matter. Any one familiar with the fire records of New York City will tell you that the greatest number of arson cases occurs when the moon is full. But does this mean that at the full moon *your* house will be set afire? Again,

quite a different matter. In other words, there are things that can be predicted with reasonable accuracy of groups of individuals, things which however cannot be predicted with any accuracy of individuals themselves. I can predict of a freshman class that most of its members will believe vehemently in the existence of God and the immortality of the soul; and I can also predict that a large percentage of them will during their college careers go through periods of atheism and materialism. But does this hold for the individual freshman? We know that it does not.

Men do not mature in their minds with the same regularity and uniformity that they do in their bodies. We can say of both Johnny Smith and Sammy Jones that at a certain time in their lives they will grow hair on their faces. But we cannot say that at a certain point in their thinking they will be Platonists. We can say of both that they will increase in stature and endurance up to a certain time and that afterward their stature and endurance will decrease. We cannot say of them that up to a certain point they will welcome new ideas and that after that time they will become conservative. But we can say of a large number of young people that most of them will go through a materialistic phase, and we can say of the large number that most of them will turn conservative.

Now the history of philosophy deals with large numbers of thinkers and with general trends. Even when we are dealing with great individuals we know that to an extent they mirror the thinking of their time. No man stands entirely by himself. The phenomena of repetition which we have observed are ones which apply to large groups of thinkers, to the majority of individuals. But they are not part of the structure of the thinking of every individual.

Individuals differ. Some are clear and analytic in their thinking and couple with this a large amount of enthusiasm for their

beliefs. Such individuals will oscillate markedly in their thinking, but will make real progress in maturing with every change. And there are others who possess more enthusiasm than power of analysis, ones who fly back and forth between one extreme position and the other. They usually progress more slowly, if at all. Remember Rousseau's *Confessions?* And there are those who are analytic but restrained. They may not oscillate at all, but take a fairly steady path in one direction. Spinoza was such a thinker. The graphs of mature thinking in individuals strike an average, but they are by no means so uniform as Hegel supposed.

The Significance of History for the Logician

Why, then, seek principles of logic in the history of philosophy? It is certainly dangerous. Hegel, like most idealists, reminds us of a man who cannot tell a story without attaching a moral to it. As we have said earlier, logic should be to thinking what arithmetic is to finance: it should express the structure involved in *any* transaction. To mix metaphysics with logic is like putting a blackmail note in a pay envelope. Logic should express the symbols of the structure of thinking, regardless of consideration as to where that thinking leads.

Hegel need not hang his head. He has not been the only sinner. The Aristotelian preoccupation with an analysis of thinking into *subjects* and *predicates* had very important metaphysical implications which Aristotle himself did not fail to draw and which ruled the thinking of men for almost two thousand years. And the recent mathematical logic has led to exactly the same error. Even though confronted by the horrible example of the Aristotelians, and striving consciously against repeating the mistake, it did not refrain from drawing morals. This newer logic uses *terms* and *relations*, as we have seen, and yields to

the temptation of moving from this analysis to a metaphysical doctrine of external relations. It is not difficult to show the falsity of this move.[1] We must keep metaphysics out of logic. Indeed the logician who mistrusts metaphysics is a better logician for it.

But if the logician can approach the history of philosophy with logic in mind, and keep his head, so much the better. The history of philosophy is to the logic of maturing thought what a concrete problem is to arithmetic. If it takes three men three days to build three boats, how long will it take one man to build one boat? A schoolboy studying arithmetic would have a difficult time if he did not work with concrete illustrations of his principles. And, similarly, the logician needs concrete examples of maturing thought.

The history of philosophy supplies just such illustrations for the logic of maturing thought. It would be as unwise to neglect this illustrative material as it would be to make a schoolboy study arithmetic without having a chance to apply what he has learned. The student of logic has begun to mature, but he is at an early stage in a process which, in the history of man's thinking, has been through many stages. Hence one reason for linking logic with the history of philosophy. In the single example we have studied in detail, the structure of maturing thought has been amply exemplified. And there are countless other illustrations that might have been employed.

The danger, of course, is that too much attention will be given to the content of the example. At no point in the history of philosophy could it be argued that that specifically was the way in which thinking *must* mature. There are other mature interpretations of nature in addition to that of Democritus, for example. Consider Leibniz and Bergson. To take such a position would be like saying that every three men can build three

[1] See my *Classical and Relational Logic*, Phil. Rev., Vol. XLV, 3, pp. 297-303.

boats in three days, just because the one example studied employed those figures as its content. It is not the specific answers given by thinkers from Thales to Democritus that count, it is the *process* through which the development of the idea went.

Those Repetitions Again

Hegel was among the first to take *moving* pictures of the structure of thinking. Give credit where credit is due. But when he came to the history of philosophy he was lured by the repetitions we have noted into giving an altogether too neat picture of maturing thought. Like most moving pictures, it always had a happy ending. Men would mature by thinking the way Hegel did! The villain was as black as night, and he was always foiled. Hegel himself was the hero. But the real world is not like the movie world. And the world of thinking is not as neat as the Hegelian description would make it appear.

Nevertheless, those elements of repetition, while not structural, are interesting. The oscillation between two extremes, which is often found in the history of philosophy, also occurs frequently in the thinking of the individual. It shows one of the ways in which in fact thinking does mature, one of the normal ways. And since most of us are normal, we shall profit in our thinking by knowing about it. If you are anxious to learn how to swim, you will find it profitable to work on the over-hand stroke. Not that you *must* swim the over-hand stroke. Perhaps you will be a side stroke swimmer. But the over-hand is one way of getting ahead in the water.

And the same general remark may be made about the second type of repetition, that in which men continue on higher and higher plateaus to solve their problems in the same general manner. Most of us keep meeting the same fundamental problems on higher and higher levels of understanding. The same

major issues are encountered again and again, because they are essential to the problem on whatever level of maturity it is attacked. To know these issues cannot but help the individual who is seeking his own maturity of thought. It does not by any means follow that the answers he gives will correspond exactly, or even remotely, to those given by the actual men he studies. The understanding of the issues will stimulate his thinking, but need not dictate his personal solutions.

The third type of repetition, that in which a cycle of maturing thought is described, is the most significant of all, and the most dangerous. It is no more a strict logical principle than either of the other two. It, also, tells only of a general trend in most thinkers. It would be completely false to argue, for example, that the later a man appears in the history of philosophy the more mature is his thinking. In fact it seems to be as true of philosophy as it is of music that the most mature figures appear in the middle of a period. To measure philosophic maturity by centuries would be as false as to say that any twentieth century scientist is a greater scientist than any Renaissance scientist. But science does mature. And so does philosophy. This point is so important that it merits special consideration.

Five Mature but Conflicting Philosophies of Life

We must go back for a moment to the structure of deductive systems. When you ascend a flight of stairs, the level tread is just as important as the difference in height between two steps. We cannot climb without level steps any more than the Alpine climber can ascend an ice wall without cutting foot-holes. And, similarly, the logic of non-contradiction and of analysis is just as important in its way as the logic of contradiction and synthesis is in its own.

One of the things we learned when we discussed deductive systems was the primary importance of initial postulates. On any given plateau of his thinking a man will have worked out more or less adequately a deductive system based on the assumptions which he considers at the time to be the most important. He will insist on a philosophic doctrine that is capable of being analyzed and which is self-consistent. The very discomfort of the intermediate periods of uncertainty is evidence of man's search for a consistent deductive system (a plateau) which will express the whole of his thinking at a given time. In working out this system the initial postulates are the all-important foundation.

Over a period of time it is to be expected that, taking advantage of what has gone before, men will find more and more adequate fundamental assumptions. As the philosophy of a period progresses one may well expect to find emerging a limited number of carefully considered and well thought-out foundations for a philosophy of life. In other words, there will be general trends in postulate sets. Toward the end of a period one would expect to find the most intelligent possibilities all present, gathered along the way, and given their best expression. The emergence of these major philosophies of life is one of the phenomena made clear in any comprehensive study of the history of philosophy. And it is reasonable to expect that the individual who is made aware of these general trends, these mature developments, will receive a stimulus toward his own thought development for which there can be no substitute.

There are at least five major conflicting philosophies which rule the contemporary field, five which merit open-minded consideration:

(1) *Negativism*, the doctrine enunciated by Rousseau and carried on by such men as Thoreau, Tolstoi and Gandhi. This is the philosophy of mystical inner life. It distrusts intellectual

efforts and material considerations. It preaches the simple life and condemns civilization. It believes that man is innately good and asserts the purpose of life to be absolutely free individual self-expression.

(2) *Platonism,* the doctrine which believes in the reality of eternal and unchanging ideals, such as Truth, Beauty and Goodness. It is the enemy of materialism and of bodily pleasure. Most often it expresses, as in St. Paul, the idea that man is innately bad and must be redeemed through discipline. In its religious phases it finds great support in emotion, but believes fundamentally that the path of reason is the path of knowledge. Matthew Arnold offers a striking example of such a doctrine.

(3) *Materialism,* the doctrine which believes in the exclusive reality of the physical. It is most often the philosophy of science and places man as an infinitesimal particle in the universe, moved by the laws of nature and without freedom of choice. It also believes in the intellect as the instrument of wisdom, the intellect experimenting in the laboratory. This is the philosophy of most modern psychology. Sometimes it is called Naturalism.

(4) *Idealism,* the doctrine which believes in the reality of man's spirit. Man by his thinking is creative of art, religion and science. Thinking creates the world, hence man is at the center of his universe and is the measure of all things. This philosophy is admittedly and primarily an attempt to reconcile earlier positions by placing them in a dialectic relation to one another. In its less formal moments this point of view is known as Humanism.

(5) *Pragmatism,* the doctrine which believes that only the practical problem is significant. That is true which works: that is good which leads to happiness. This is the most recent attempt to reconcile earlier positions, by showing that they all have elements of truth and that these elements may be combined eclec-

tically. It treats each individual problem on its own merits, and does not attempt to build a metaphysical system. It is for this reason that some do not include pragmatism among philosophies, but treat it as the denial of philosophy and the ennobling of sheer common sense.

These are major types of philosophy. The individual thinker may fit neatly into one of the classifications, or he may devise a system of his own which is unique. Bergson, for example, is something of a pragmatist, something of a negativist, and something of a Platonist. The vitalism which he represents bids fair to become a new type of philosophy taking a place with the others. Whitehead is another who will not fit into this classification. He also has worked out a philosophy which may take a place as a new major system. But in dealing with most philosophers, past and present, this classification will be found helpful.

History's Most Important Contribution to the Logic of Maturing Thought

It is right here that the history of philosophy plays its unique rôle. One of the things which any one with intellectual curiosity is interested in doing is to know intimately the major trends of philosophic thought, the most important positions that our great thinkers have taken. When he has come to know these interpretations sympathetically and understandingly he will be ready to choose intelligently his own point of view. This is the function of the study of the history of philosophy, to help students reach maturity of thought by showing in mature examples what the most significant possibilities are.

It is something like choosing a wife. If you go constantly with the first girl you meet you will not know her very well because you do not have a knowledge of other girls whereby to judge her. Meet many girls, meet different types of girl; then you

will be more sure of choosing wisely. You may come back to your high school sweetheart, but you will have a more mature appreciation of her. It would be an inestimable privilege to gather all of the great thinkers of our time together and converse with them about philosophic problems; Dewey, Bergson, Russell, Whitehead, Croce and Gentile. It would be an even greater privilege to talk to Plato and Aristotle and Descartes and Spinoza and Locke and Hume and Kant and Hegel. There could be no greater intellectual stimulus. This is the stimulus offered by the history of philosophy, one which is unique in the profoundness of its opportunities and in the delights of its friendships.

The All-Important Difference between the What and the How in Philosophy

The point cannot be reiterated too often. There are many different but mature philosophies of life. The point is not *what* you think, it is *how* maturely you think it. This is the great lesson of this aspect of logic, and also the great lesson of the history of philosophy. They tell the same story.

Maturity has to do with the structure and not the content of our thinking. You can take the fundamental assumptions of the modern materialist and develop from them a philosophy which will be both consistent and adequate. You can do the same with the fundamental assumptions of Christianity, or with those of the Idealist. One of the things that both the history of philosophy and logic should teach is that there are several conflicting yet significant and important ways of describing the same series of events.

As I sit writing these pages I see out of my window an unending series of waves breaking on a beach. The poet standing on this shore would seek opportunity for free self-expression

and might describe these waves lyrically, the "white seahorses" of Arnold's *The Forsaken Merman*. Many hundreds of years ago the Psalmist walked along a shore like this and heard the voice of God on the waters. A scientist might give a description which would include the mechanics of the sine curve and the chemistry of sea water. An Idealist would find in this situation an idea created by man's thinking spirit. The pragmatist would launch a boat in these waves or swim through them to know them. Each sees something different and each makes a different interpretation. Which is true? All are. The essential thing about fundamental assumptions is that they are fundamental, that any attempt to argue them or to get behind them leads only to other fundamental assumptions. Philosophy will always be dependent upon assumptions; this we know from our study of deductive systems. And assumptions by their nature cannot be argued.

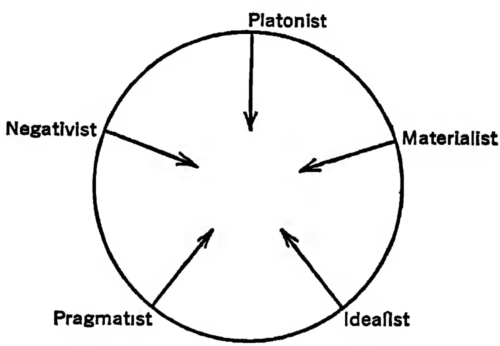
The Saving Factor in a Situation that Would Otherwise be Hopeless

Are you alarmed by this situation? It does seem critical. We find ourselves arguing that it is impossible to choose among a number of fundamentally conflicting philosophies of life. Each is profound. Each is valid. Each is adequate. What to do? The situation would be without hope and lead to complete skepticism were it not for one fact. And in this fact is contained the most significant lesson in the entire history of philosophy, the lesson toward which every study of philosophy must lead if it is to perform its most vital function. Without it, incidentally, there would be no point in studying logic. Let me ask you a leading question.

If you had an Invisible Cloak which would enable you to follow a philosopher around unseen from sunrise to sunset,

would you be able to tell whether he was a materialist or an idealist, a Platonist or a pragmatist? This may be heresy, but I am not altogether certain that you would—if the philosopher knows his business. Something curious happens to philosophies as they mature. History tells us that as the philosophy of the Epicureans developed it took more and more into account the function of reason in evaluating pleasures. And we learn that the greater maturity of the Stoic tendency took more regard for the sensual. The two come closer and closer together, until a good Epicurean is difficult to distinguish from a good Stoic. And in modern philosophy the same thing has happened. Would you distinguish between a Platonist and a materialist? There is nothing like a good understanding of the chemistry of the human body in leading one to the Christian virtues of tolerance and forgiveness. And a thorough understanding of religion places it in its natural setting. The more mature each becomes in his own philosophy the closer together the two philosophies find themselves.

This heresy may be formulated into a theory which is at bottom a logical principle. Philosophers are like men standing in a circle contemplating reality. As they become more and more mature in their thinking they move closer toward the center and toward one another:



Their fundamental assumptions will remain far apart, and the words they employ may be quite different, but in their living they will become more and more alike. It would be interesting to list the propositions to which all thinking men, whatever their philosophies, will agree. Will they not agree, for example, that hate and intolerance and pride are bad? Will they not come to believe in the simple life, and see the folly of piling up material goods? Will they not have a fundamental belief in the importance of reason? Such propositions as these are the measures of mature thinking whatever the philosophic doctrine.

Logic Is the Key to Intellectual Maturity

The danger in thinking is not the danger of a particular philosophy. Neither communism nor capitalism, nor atheism nor Christianity, is in itself dangerous. The danger is in being an *immature* communist or capitalist, atheist or Christian. Every philosophy is dangerous in its immature stages and every philosophy is wise in its maturity. In the early stages of one's thinking one is apt to be inconsistent. There is a danger. In the early stages one may be guided by expediency rather than principle. There is a danger. In the early stages one goes to belligerent extremes. Again, a danger. The danger is in the wishful "thinking" of the Nordic who rewrites all of history in terms of Nordic supremacy, in the nation which is trying to prove its "friendliness" by waging an undeclared war, in the narrowness of the man or woman who rushes to get a marriage license before the law which will insure freedom from social disease is enforced, in the manufacturer who will pay for the suppression of research which shows that his breakfast food is not healthful; in short in all action based on the force of money or arms or blood aristocracy or tradition or political party, rather than the

force of logic and reason. These are the signs of immaturity and there is no greater danger in the world to-day.

The help against these things is persistent logical thinking which will understand the importance of valid reasoning, which is ready at every moment to recognize inconsistencies and move forward to more and more mature thinking, and which will realize that there are no final truths and hence no end to the search for knowledge and understanding.

Appendix

TWENTY-FIVE BRAIN TEASERS

On Which to Test Your Logical Ability

Omitting the more simple catch questions, such as "Should a man be allowed to marry his widow's sister?" and "If it takes a clock 4 seconds to strike four o'clock, how long will it take it to strike eight o'clock?" (the answer is neither *8 seconds* nor *4 hours*), this list is intended to include one of each of the more prominent types of intellectual puzzle involving logical manipulations.

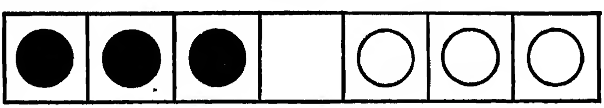
1. Arrange 13 pennies in 9 straight rows, 4 in each row.
2. The ages of a man and his wife are together 98. He is twice the age that she was when he was the age she is today. What are their ages?
3. Can you decipher this cryptogram?

ABCDE DF GIH JKG BL
MBDOM PKBOM PDGI
EBOLDRHOEH.

4. A boy in college sent his father the following message in which the letters stand for numbers. How much money did he want?

$$\begin{array}{r} \text{S E N D} \\ +\,\text{M O R E} \\ \hline \text{M O N E Y} \end{array}$$

5. Six checker men are arranged as follows:



The black men can move only to the right, and the white men only to the left. A man can move one square if the adjacent one is unoccupied, two squares by jumping if the adjacent square is occupied by a man of the opposite color. Put the black men where the white ones are, and the white ones where the black ones are.

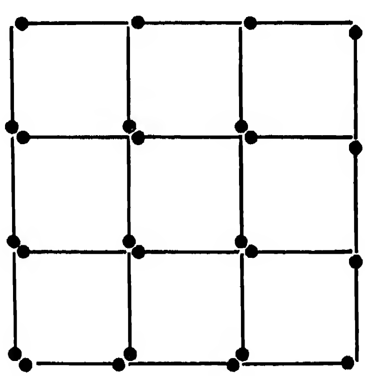
6. A man took 100 children on a picnic. He bought 75 sundaes for the girls and 25 ice cream cones for the boys, noting that the sundaes cost 4¢ each more than the cones. If his total bill was ten dollars, how much should he collect from the boys for their cones?
7. Three pretty girls and their three jealous fiancées want to get to a distant movie on a motorcycle which will hold only two. How get the whole company there, never leaving a girl in the company of a boy unless her fiancé is present?
8. A ship of the Lunard-Blue Star Line sails each noon from New York for Cherbourg, and another sails at the same hour each day from Cherbourg to New York. Each ship takes exactly a week to make the crossing. If an officer of the line takes passage on one of these boats, how many belonging to his line will he meet during the passage?
9. Brave men tell only truths: cowards tell only lies. Three men meet on the street. The first identifies himself to the second, who turns to the third, saying, "He says he is a brave man." The third replies: "He is not a brave man, he is a coward." How many brave men and how many cowards are present?
10. It is sixty miles from Springfield to New Haven. The mayor of Springfield starts for New Haven on a bicycle at the rate of 10 miles per hour. At the same time the mayor of New Haven starts for Springfield on a motorcycle at the rate of 20 miles per hour. Just as they start a

fly leaves the nose of the mayor of Springfield and flies at the rate of 30 miles per hour toward the nose of the other mayor and, reaching it, starts back toward the nose of the first mayor at the same speed. If the fly keeps flying between the mayoral noses until he is crushed to death when the on-coming mayoral noses meet, how far has he flown?

11. A board has three holes cut in it: (1) a circle of 6 inch diameter, (2) a square 6 inches on a side, and (3) an isosceles triangle of altitude and base 6 inches. What is the shape of a rigid body which can pass completely through each of the holes yet fill each completely as it passes through?

12. A Russian woman fills her samovar with 24 gills of tea. She wants to divide the tea equally among three people, and has in her possession three containers that hold 5, 11, and 13 gills. How can she do it?

13. Twenty-four matches are arranged thus:

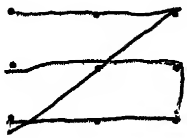


Take away 8 matches in such a way as to leave two large squares and one small one.

14. A man rowing upstream passes two mile markers. At the second marker his hat blows into the river. Ten minutes

after leaving the second marker he turns around and rows after the hat, and picks it up at the first marker. What is the speed of the current?

15. Without retracing a line draw four consecutive straight lines so that they pass through all of these nine dots.

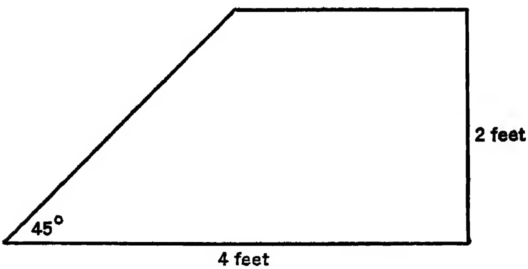


16. One cup is half full of tea, another half full of water. A tablespoon of tea is taken from the first and placed in the second; then a tablespoon of the resulting mixture is taken from the second and placed in the first. Which was greater, the amount of tea taken from the first cup or the amount of water taken from the second?

17. What is the smallest number of queens that can be placed on a chess board so as to command all squares?

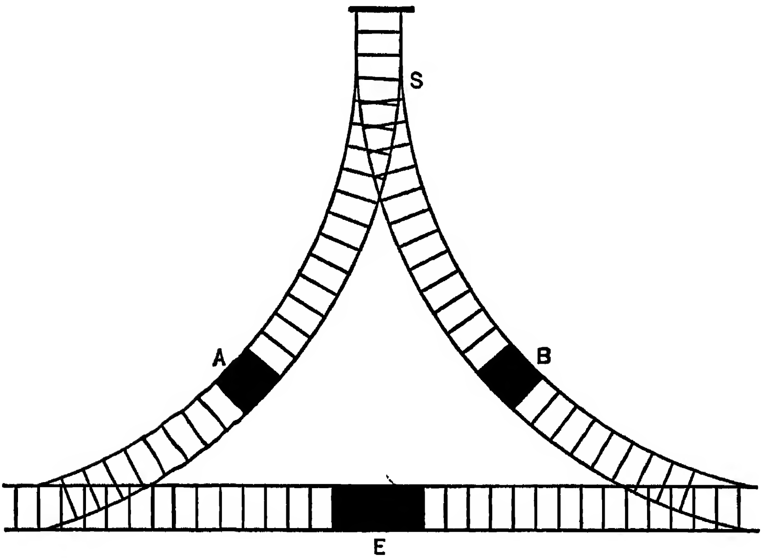
18. At a Christmas bazaar there were one hundred men, women and children. The men spent \$10 each, the women \$3 each, and the children 50¢ each. If the bazaar took in \$100, how many men, how many women, and how many children were present?

19. Divide the following area into four identical parts.



20. Mr. C commutes from the city on a train of uniform velocity and is met at the station by his chauffeur, who drives at a uniform velocity. One day he arrived at the station an hour early and started to walk home. He was picked up by his chauffeur and reached home ten minutes earlier than usual. How long did he walk?

21. There are two freight cars and an engine arranged as in the figure. The engine, E, can run up either siding but must come back the same way because it is too large to pass the switch at S. How can it interchange the cars, A and B, without using a flying switch?



22. Two pipes lead water into a swimming pool. Either of the two pipes can fill the pool in three hours. When the pool is full it can be drained in six hours. How long will it take to fill the pool if both pipes are pouring water into it and the drain is open?

23. A clownish friend places smudges on the foreheads of his

three sleeping friends. When they awaken, they all begin to laugh at each other but, being very polite, refrain from pointing. After a few moments one of the three takes out his handkerchief and wipes the smudge off his own forehead. By what process of reasoning did he become certain that his forehead was smudged?

24. A man went into a hardware store and bought a $4 reading lamp, giving the clerk a ten-dollar bill. The clerk could not change the bill, so he took it to the corner drugstore and got ten one-dollar bills in exchange. When the transaction was complete and the customer had left, the drugstore manager came running to the hardware store to say that the ten-dollar bill was counterfeit. Of course the manager of the hardware store gave him a good ten-dollar bill immediately. What was the loss to the hardware store?

25. There are three musicians; a violinist, a cellist, and a pianist. Each is the father of a grown-up son. The sons' names are Brown, Town, and Gown.

a. The cellist and Town, Jr. are six feet tall.
b. The pianist is five feet tall.
c. Gown, Jr. is six inches shorter than Town, Jr.
d. The violinist is five feet nine.
e. The violinist has exactly one third as many Victor records as that man (among the other five) who is nearest his height.
f. The pianist's son has 313 orchestral records and 409 vocal records.
g. Brown, Jr.'s father has more false teeth than the cellist.

What is the name of the violinist?

Dividend for Digit Demons:

How far up the scale of numbers (not skipping any) can you go by expressing each number in terms of *four* 4's? The only symbols at your disposal are the arithmetic signs for addition (+), subtraction (—), multiplication (×), and division (÷), parentheses, and the decimal point. E.g.

$$(4 + 4) \div (4 + 4) = 1$$
$$(4 \times 4) \div (4 + 4) = 2$$
$$(4 + 4 + 4) \div 4 = 3$$

PROBLEMS

Problems are of the same importance to logic that exercises are to mathematics. Both are overrated. Some people have the idea that mental exercises are like setting-up exercises, that they develop mental muscles. Nothing is more misleading. You can do exercises on syllogisms, inductive methods, and deductive systems for days without end and not think any more clearly when you stop.

These exercises will serve you best as a measure of your understanding of the structure of thinking. But even here there is a catch. If you understand the material of this book you will be able to do the exercises easily and correctly. But it does not follow that if you can do the exercises correctly you understand the material! You must remember as well as I do the high school experience of getting 100% on an assignment without having the slightest comprehension of what you did. The measure is given, of course, in how you do the exercises and not in whether or not you produce the correct answers. Do you understand the problem or do you handle it by-guess-and-by-gorry?

Introduction

1. Discuss the similarity of structure in the following pairs of arguments:

   a. His appearance here would imply the victory of the government forces. But the government forces have been defeated, so we need not look for him to come.

      If you press this button you will hear a Brazilian radio station. You are actually listening to Mexico City, so you must have pressed another button.

b. No Romans carried watches. Some Romans were conquerors. Therefore some conquerors did not carry watches.

All dogs are excluded from this park, but many dogs are friendly, and hence some friendly animals are kept out.

c. I came before he did, and he came before the door was opened. Hence I was also there too early.

These stories must be entertaining, for all of his stories are entertaining.

2. Discuss fully the dissimilarity of the structures of the following arguments. Which are valid? Why?

a. The skipper is east of the mate.
The mate is east of the cabin-boy.
———
The skipper is east of the cabin-boy.

The skipper is the friend of the mate.
The mate is the friend of the cabin-boy.
———
The skipper is the friend of the cabin-boy.

b. Farmers are prospering. Therefore this farmer is prospering.

Farmers can vote. Therefore this farmer can vote.

3. Find two arguments in the morning newspaper and show clearly the difference between the problem of *truth* and the problem of *validity* as regards each.

4. Find ten words which have entered the language since 1900, and discuss their usefulness and their relation to older words.

5. Find three words whose meanings have changed significantly since Shakespeare's time, and trace the change.

6. Give *your own* definitions of the following:

democracy	trunk	logic
student	courage	machine
circle	man	freedom

yourself as an individual

7. Discuss the adequacy of the following definitions:

   a. A moving-picture is a picture that moves.
   b. Woman is a weaker man.
   c. A square is a four-sided figure the sides of which are parallel.
   d. This is a shaving device that does away with soap and brush.
   e. Man is a social animal.
   f. Duty is the stern daughter of the voice of God.
   g. Patriotism is the last refuge of a scoundrel.
   h. Dancing is a measured leaping, tripping or stepping in unison with music or rhythmic beats.
   i. Soldiers are brave men.
   j. A politician is one that would circumvent God.

8. Identify and expose the fallacies in the following arguments:

   a. Wake up, men. Be not like dumb driven cattle. In this land of the free and the brave no man should be told what crops to plant.
   b. Gentlemen of the jury, you should be moved by sympathy far more than by condemnation in this case. All his life this unfortunate man has been connected with a phase of finance which tempts man to make his fortune by shrewdness and luck. I ask a verdict of "Not Guilty."
   c. The women of this country are unalterably opposed to such a measure, hence she will vote against it. Is she not a woman?
   d. Education cannot prepare men and women for newspaper work. To try to do so would be like trying to teach them how to get married and like it. It cannot be done.
   e. We have gotten along for a hundred and fifty years without governmental interference of this sort, and at the same time the American working man has achieved a luxurious living such as has never been approached elsewhere. We must oppose this attempt to interfere.
   f. If you love your country you will give her sons.
   g. Inasmuch as the President is head of the Navy, is it not strange that, in his business, he goes so contrary to the advice handed out to heads of other enterprises?

h. Planned parenthood is paganism gone pseudo-scientific, subjecting parenthood to sex, and, far from making for democracy, makes a man a sexual totalitarian or a brut-talitarian.
i. If a child has a new horn he wants to blow it; if a child has a new tricycle he wants to ride it; if a nation has a new navy it wants to use it.
j. A high tariff has helped infant industries grow strong, it has brought money into the treasury, it has increased employment. A high tariff is a good thing.
k. Birth control brings a higher standard of living, for wherever it is practiced the standard is higher.
l. How can you argue that you are a free agent when it is impossible to prove that you are?
m. In the course of my work I have dissected more than a thousand human bodies, but never found the human soul. Man has no soul.
n. Do not expect financial aid from him. He comes from the poorest town in the whole state.
o. Can this be the workingman's candidate? He lives on Park Avenue, he is driven around in a limousine by a chauffeur, he sends his children to a rich man's school, he never got his hands dirty in his life. You must not support him!
p. There is not a man on the faculty who is not a good teacher and an outstanding scholar. You are sure to get a fine liberal education if you come here.
q. A big strawberry is a super-strawberry; a big liner is a superliner; a big spectacle is a super-spectacle; a big band is a super-band; hence a big man is a superman.
r. The League of Nations, founded to maintain international law and order, has been in existence for twenty years; yet during that time Japan has invaded China, Italy has invaded Ethiopia, and Germany has repeatedly broken the Treaty of Versailles. It is useless to give our support to an organization so ineffectual.
s. Every man in this orchestra has been picked for his outstanding merit as an instrumentalist, and hence this is the finest orchestra in the country.

t. Education equips a man to meet intelligently and understandingly the problems of life. A trip around the world is an education. Therefore, a trip around the world equips a man to meet intelligently and understandingly the problems of life.

u. As your father I refuse to allow you to go around with that young man. He belongs to the most disreputable fraternity on campus.

v. At your birth Venus and Mars were in favorable positions, and you will be successful in love and war.

w. See this mother, her face lined with care; see these children, the fruit of years of love and devotion; picture their happy family circle, the pitter-patter of little feet, the laugh of the innocent child. Could you think this woman, this symbol of all that is best in her sex, one who could have perpetrated such a dastardly crime. It is preposterous!

x. Any man who takes property from another is a thief. The move toward government ownership is nothing short of an attempt to take a man's property from him. It is thievery!

y. The priests went out to the fields and they blessed the land and the tools of husbandry. And that summer the crops were rich and plentiful.

z. Polygamy is a crime, yet polgamy is encouraged by the laws of Arabia. Hence the laws of Arabia encourage crime.

Chapter One

1. How many types of structure can you find in:

"Lewd did I live & evil I did dwel"

2. Show that the following pairs of objects are alike in possessing structure:

a watch	checkers
a house	bridge
a symphony	a sonnet
a painting	a circle

3. By means of illustrations show the importance of the logical distinction between *classes* and *individuals*.
4. Distinguish clearly between (1) *logical* terms, and terms in other structures, and (2) *logical* relations, and relations in other structures.
5. Classify the following relations according to *two* different types of analysis; noting that some may be interpreted in more than one way:

between	companion of
larger than	joined in matrimony
brother of	equals
fattest of	played a quartet with
chauffeur of	purrs
repeats	diametrically opposite to

6. Give examples of relations which are:

   transitive but asymmetrical
   intransitive but symmetrical
   transitive and symmetrical
   intransitive and asymmetrical

7. Bring the following propositions into *strict* class-analysis form, employing each time "included in" or "excluded from":

   a. Only boys know how to whistle.
   b. Some books are to be tasted.
   c. No beer, no work.
   d. All is lost save honor.
   e. Somewhere in this favored land the sun is shining bright.
   f. Man was born free and is everywhere in chains.
   g. Nothing is sacred.
   h. There is something in this more than natural.
   i. Every man shall bear his own burden.
   j. No man but a blockhead ever wrote except for money.
   k. A thing of beauty is a joy forever.

l. All the brothers were valiant, and all the sisters virtuous.
m. Whatsoever a man soweth, that shall he also reap.
n. There is nothing either good or bad, but thinking makes it so.
o. There are those who do not have homes.

8. What relations can you establish within the following pairs of propositions?

a. This is his second offense.
This is his third offense.

b. The dog's name is Fido.
The dog's name is not Fido.

c. Every day is a holiday for him.
Few days are holidays for him.

d. Some wear their hearts on their sleeves.
Some do not wear their hearts on their sleeves.

e. Every man has his price.
This man does not have his price.

f. All things change.
Nothing changes.

9. If the first proposition in this group is true, state of the others whether they are true, false or doubtful.

a. No philosopher is practical.
b. Some philosophers are impractical.
c. All philosophers are impractical.
d. Some philosophers are not practical.
e. All philosophers are practical.
f. This philosopher is practical.
g. Some non-philosophers are practical.
h. All non-philosophers are not practical.
i. Some philosophers are practical.
j. This non-philosopher is practical.
k. No philosopher is not impractical.

10. By way of examples, show clearly the difference between (1) contraries and contradictories, and (2) contraries and sub-contraries.
11. By way of examples, show the difference between *formal* and *material* (1) contraries, and (2) sub-contraries.
12. Convert those of the following propositions that can be converted:

    a. Death and taxation are unavoidable.
    b. Few men are desperate characters.
    c. Some swans are not white.
    d. All Christians are excluded from Mecca.
    e. No one has ever seen a fairy.
    f. Every schoolboy knows how to polish apples.
    g. Some modern artists are not honest.
    h. Some windships still sail the seas.

13. Can you work out a diagram of your own which will illustrate clearly the true-false-doubtful relationships among A, E, I, and O propositions?

Chapters Two and Three

1. Define the syllogism. Find *three* arguments in books other than logic books, put them into a syllogistic form, and show how they fulfil the definition.
2. Identify:

middle term	major premise
subject term	minor premise
predicate term	

What parts do these elements play in the syllogism? Label the terms and premises in the three examples of Question 1.

3. Construct syllogisms with the following names:

   1st figure—Darii, Ferio
   2nd figure—Camestres, Baroco
   3rd figure—Darapti, Datisi, Felapton
   4th figure—Bramantip, Camenes, Fresison

4. What rules are violated by syllogisms based on the following groups of three propositions, the last proposition in each case being the conclusion?

   IAA EEE OAE AEI

5. What rules are violated by the following syllogisms? Give the technical name of the fallacy. Can a valid conclusion be drawn from the premises as they stand? Can one be drawn if they are reversed? If so, what is the name of the valid syllogism and what is the conclusion?

   a. All churches are houses of worship.
   This is a house of worship.
   ———
   This is a church.

   b. All little boys are devils.
   Some devils carry pitchforks.
   ———
   Some little boys carry pitchforks.

   c. Some games are games of skill.
   No games of skill are easy to learn.
   ———
   Some things easy to learn are not games.

   d. No witches are real.
   All who are real are tangible.
   ———
   None who are tangible are witches.

6. Express the four syllogisms in Question 5 in symbols, using:

   (1) classes, "all" or "some," and "included in" or "excluded from."
   (2) classes, "not," and signs for equality or inequality (see the discussion of the antilogism).

7. Deduce all possible conclusions from the following pairs of premises. Name each syllogism:

a. All who own property should have the vote.
Some women own property.

b. All who deny freedom of speech are afraid of the truth.
All who deny freedom of speech are dictators.

c. Some nations will not arbitrate international differences.
All who will not arbitrate international differences must be handled by force of arms.

d. No right-thinking man is governed by his emotions.
All who welcome war are governed by their emotions.

8. Can you explain why in the class analysis of propositions "most men" and "few men" must be treated alike, both being symbolized as "some men"?

9. Select three sets of premises to support each of the following conclusions, such that none of the three arguments for any one conclusion will belong to the same figure.

a. Some children need to be spanked.
b. Some spinsters are not neurotic.
c. No man is a hero to his valet.

10. Show why, in both of these cases, if the first of the two syllogisms is valid the second must also be valid.

a. None who are honest are primarily interested in increasing sales.
All who write advertisements are primarily interested in increasing sales.
___
None who write advertisements are honest.

All who write advertisements are primarily interested in increasing sales.
None who are primarily interested in increasing sales are honest.
___
None who are honest write advertisements.

b. All who do not produce economic goods are parasites.
Some wives do not produce economic goods.
―――――――――――――――
Some wives are parasites.

All who do not produce economic goods are parasites.
All who do not produce economic goods are wives.
―――――――――――――――
Some wives are parasites.

11. Discuss *fully* the problem of the validity of the following argument:

All circuses are animal shows.
All animal shows delight children.
―――――――――――――――
Some things that delight children are circuses.

12. Test the following arguments by using these tests: (1) circles, (2) Latin names, (3) the six rules of the syllogism, (4) the antilogisms:

He must be happy, for all who are happy are fat and he certainly is fat.

Some veterans are lobbyists, hence some veterans are menaces to good government for all lobbyists menace good government.

13. Show that the *fifth* of the rules of the syllogism holds for all valid syllogisms.
14. Why is the diagram for Figure I syllogisms so similar to to the diagram for Figure II syllogisms? Draw all possible parallels between the two figures.
15. Can you work out a diagram of the twenty-four important syllogistic arguments which will show clearly *all* of the relationships between them that are discussed in Chapters Two and Three?

Chapter Four

1. Analyze the following arguments (1) atomically and (2) molecularly:

a. All men who wear tight-fitting hats get prematurely bald, and they also wear lemon-colored gloves, so we can be sure that some men who wear lemon-colored gloves are prematurely bald.

b. I shall have to practise regularly if I am to play the piano well, but alas I cannot find time for regular practice and it will be well for me to give up my aspirations.

c. That is a real ghost or I am a fish-faced baboon. It *is* a real ghost.

2. Test the following arguments. If any are invalid name the fallacy committed.

a. If God created the world it should show order and design. The fact that it does show order and design indicates clearly that He did create it.

b. If that is the best you can do, then I must dispense with your services. You admit that you cannot do better. Good-by!

c. The witness tells us that he did not see the accident because either his back was turned or he was attending to something else. He must have been attending to something else, because we know that his back was not turned.

d. The dress I am looking for is either Alice blue or it has a large gold buckle. I remember the buckle distinctly, so do not bother to look for a blue dress.

3. What conclusions, if any, may be drawn from the following premises?

a. If I take the low road I will get to Scotland before you.
I take the high road.

b. If you are Greta Garbo, then I am Walter Winchell.
I am not Walter Winchell.

c. This boat is either improperly designed or badly built.
It is badly built.

4. What is the structural difference between the following pairs of propositions?

   a. I want him captured either dead or alive.
   Please serve the fish either broiled or with egg sauce.

   b. It is a genuine article only if it bears the trademark.
   If that is a bear we had better start running.

5. Transform the following arguments into (1) implicative, and (2) disjunctive form:

   a. None who can reason can be fooled all of the time.
   All men can reason.
   ___
   No men can be fooled all of the time.

   b. All things that keep perfect time are stationary.
   No wrist watch is stationary.
   ___
   No wrist watch keeps perfect time.

   c. All who have hobbies are happy.
   All who have hobbies are men.
   ___
   Some men are happy.

   d. Some who are graceful are young people.
   All young people dance.
   ___
   Some who dance are graceful.

6. Transform the following arguments into syllogisms.

   a. If I eat too much ice cream, then I shall be sick.
   I ate too much ice cream.
   ___
   I shall be sick.

   b. Either you stay in bed or you get a spanking.
   You do not stay in bed.
   ___
   You get a spanking.

7. Give a valid complex implicative argument in which the consequent is denied. Compare its structure with that of the corresponding simple implicative argument.
8. Express symbolically the structure of a valid complex disjunctive argument, and show clearly the meaning of the disjunctions involved.
9. How would you meet the challenges of the following dilemmas? Can they be met in more than one way? If so, how?

   a. If I take this report card home I shall have an unpleasant accounting with my father; if I do not take it home I shall have an equally unhappy accounting with my teacher. But I must either take it home or not take it home. Woe is me!
   b. If democracies allow freedom of speech, they are permitting the incursions of fascism; if they do not allow freedom of speech they are allowing the same incursions. But they must do one thing or the other, so no matter what they do fascism will enter.
   c. If man is to survive on this planet he must either control the advances of civilization, or he must turn the clock of civilization backward. But he can neither control the advances of civilization nor turn back the clock. Hence he cannot survive.
   d. If you tell me that you have never broken a Commandment, I shall punish you as a liar. If you tell me that you have broken a Commandment, I shall punish you for your sin. So in either case you will be punished.

10. Take a single argument and express it as a syllogism, an implication, and a disjunction in such a way that in all three arguments one of the premises and the conclusion will be identical.

    a. What can you say about the other premise as it appears in each of the three arguments?
    b. What conclusions does a comparison of this kind suggest as regards the three types of argument?

11. Try to connect together some of the premises which belong to the surface of your thinking. At what points do you become suspicious of contradictory premises?
    Can you give definite statement to the fundamental assumptions on which any of these premises are based?

Chapter Five

1. By way of a concrete example show the importance of the contents of thinking in determining its inductive structure. Compare this situation with that in deduction.
2. Arrange the following facts in a pattern which will converge on the hypothesis that your next-door neighbor is a foreign agent.

    a. You hear the hum of many voices from your neighbor's apartment every Tuesday evening.
    b. Your neighbor subscribes to *Il Messaggero.*
    c. He carries a black leather briefcase habitually.
    d. You have seen foreign-looking strangers entering and leaving his apartment.
    e. He has just bought an expensive radio.
    f. You have seen catalogues from chemical laboratories in his mail.
    g. He is short and dark.
    h. The cleaning woman tells you that he keeps all of his drawers in his apartment locked.
    i. There is a suspicious bulge at the location of his hip-pocket.
    j. You saw him at a performance of *Aïda.*
    k. He dresses lavishly and expensively.
    l. You have seen him four times in a street café with the same strikingly beautiful blonde.
    m. He repulses your neighborly advances.
    n. He is often away from his apartment for extended periods.

o. He has a telephone, but his name is not listed in the telephone directory.
p. The windows of your apartments overlook the Brooklyn Navy Yard.
q. The day you borrowed his copy of the *Herald-Tribune* you noticed that he had worked out the daily cryptogram.

What other hypothesis or hypotheses might account for these facts?

How might you apply the principle of prediction in your investigation of this man? Show in what manner a prediction fulfilled would alter the strength of the hypothesis.

3. Organize the facts in the text about Freshman Day so that they converge on another hypothesis or other hypotheses. Reorganize the facts about Freshman Day so that they converge in another manner on the hypothesis suggested in the text.
4. Arrange the following facts so that they converge on three different hypotheses:

a. Mr. A took the midnight plane from Los Angeles to New York.
b. His father lives in New York.
c. Mr. A has often expressed a fear of riding in planes.
d. He was visited by his broker the afternoon before taking the trip.
e. He was scheduled as principal speaker at a Los Angeles Rotary Club luncheon the next afternoon.
f. He recently entertained at his house the chief pilot of one of the China Clippers.
g. His mother is very ill.
h. The weather for flying has been unusually bad, and still is.
i. Mr. A is part owner of the Empire State Building.
j. Mrs. A went to New York by train five days earlier.
k. Mr. A took with him a large heavily-packed suitcase.
l. He seemed worried before entering the plane.

m. The first thing he did on arriving in New York was to buy every available morning paper.

n. Two days before the trip Mr. A spent the afternoon at the airport inquiring about facilities for civilian instruction in flying.

5. Analyze a Sherlock Holmes story according to the pattern of convergence. Show how the pattern and the major hypothesis changed as the investigation proceeded.
6. Find in a detective story a good example of a prediction technique and show clearly its bearing on the case in question.
7. Find in a detective story a good example of a *reductio ad absurdum* technique. What, in this case, was its value?
8. Choose a detective story in which the major facts are all given early in the story and outline the major hypotheses which seem to recommend themselves as worthy of further investigation. Show how during the investigation prediction and *reductio ad absurdum* might profitably be used by the investigators.
9. Give an example of a convergence within a convergence within a convergence.
10. Write the following as arguments involving prediction, and as arguments involving *reductio ad absurdum:*

a. Columbus' speech to Ferdinand and Isabella.

b. The case for the one-time existence of the continent Atlantis.

c. The case for the New Deal.

d. The case against the New Deal.

e. The case for the entry of the U. S. into the League of Nations.

f. The case against the entry of the U. S. into the League of Nations.

g. The case against astrology.

h. The case for or against mental telepathy.

11. Look up the following scientific incidents in an encyclopedia and describe the pattern of argument involved. Show clearly the strength and limitations of the predictions involved:

   a. the discovery of the planet Neptune.
   b. the discovery of the last four elements in the atomic table.
   c. Galileo's experiment at the tower of Pisa.

12. Show the connection between the deceptive character of the argument by prediction and the fact that it involves a logical structure which is strictly invalid. Show why, nevertheless, prediction is a valuable technique.

13. The following hypotheses were at one time generally considered to be absurd. Reproduce as well as you can the argument in each case which led to these conclusions, and show structurally how it has since been overcome.

   a. that man would ever be able to fly.
   b. that a man in Paris could converse with a man in Chicago.
   c. that organisms as small as bacteria exist.
   d. that man could see the bones of his body without breaking the skin.

14. The following hypotheses are now generally considered to be absurd. Reproduce the argument in each case that leads to these conclusions, and show the possibilities for the future of this argument.

   a. that man will be able to visit the moon and return.
   b. that life will be produced in a test-tube.
   c. that a man's life can be read in the lines of his palm.
   d. that western civilization will collapse in 200 years.
   e. that men will see electrons.
   f. that non-resistance is more powerful than aggression.

Chapter Six

1. Many sailors will not leave port on a Friday: many landsmen will not walk under ladders. Trace these superstitions to probable sources and criticize the logic of their growth.
2. Cite a superstition which has a profound influence on the lives of a great many people. How would you proceed to combat it?
3. Cite two familiar cases in which the Method of Simple Enumeration

   a. has so far been adequate to the problem involved. What are the future possibilities in each case?

   b. has for a long time seemed adequate, but then suddenly failed. What factor had been overlooked?

4. What inductive method or methods would you employ in investigating each of the following problems? Why?

   a. Does coffee keep you awake at night?

   b. Why does the chandelier not light when I snap its switch?

   c. Which method is more effective in teaching arithmetic, a rigorous discipline or one based on the child's interests?

   d. Do sun spots affect radio reception?

   e. What is the secret of living to a ripe old age?

   f. What physical characteristic makes a man a great marathon runner?

   g. Does smoking impair intellectual activity?

5. Show clearly that none of the methods recommended for handling these problems is strictly infallible.
   Broaden this into a general explanation of why no experimental procedure, no matter how meticulous, can establish a generalization with finality.
6. Give a necessary cause of:

   a. the proper running of an automobile engine.

b. being hungry.
c. smoke.

Give a sufficient cause of:

a. the failure of a watch to tell time.
b. death.
c. boiling water.

7. What is the effect on the logical efficacy of the inductive methods of the fact that two or more factors may combine to produce an effect when no one of them alone will produce it?
What would be your procedure of investigating this possibility in a concrete case?
8. Cite two instances in the history of science in which faulty generalizations were "established" because the supposed cause was not carefully analyzed.
9. Illustrating your point by means of familiar laboratory procedures, show that the methods of Agreement, and Difference, and Concomitant Variations are methods of elimination.
10. Make a detailed comparison of the Joint Method of Agreement and Difference and the combined methods of Agreement and Difference.
11. You are unable to receive Station WXYZ on your radio. You do not know whether this is because of your location, the make of your set, the size of your radio, the steel frame of the building in which you live, the influence of a powerful receiving set next door, the size of your antenna, the direction of your antenna, the age of the tubes in your set, faulty wiring in the set, the strength of the signals from WXYZ, the possibility that WXYZ has ceased to function, or your own inability to tune a radio properly.

a. How would you proceed to find the cause?
b. Show how each of the following methods might be applied to this problem:

    i. simple enumeration
    ii. joint method
    iii. agreement
    iv. difference
    v. agreement and difference
    vi. concomitant variations.

12. You suspect that the reason why you are not getting good results with your new camera is that the film you are using is of poor quality. Considering all of the factors that enter into the situation, how would you proceed to test your suspicion? Name the logical method or methods employed.
13. The following experiments are famous in the history of science. Look them up in their details and explain the logic according to which each experiment was conducted.

    a. the Michelson-Morley ether experiment.
    b. Benjamin Franklin's kite experiment.
    c. Morgan's experiments with fruit flies.
    d. Pasteur's experiment with sheep.
    e. Madame Curie's experiments on the cause of radio-activity.
    f. experiments on the cause of bubonic plague.
    g. the Cuban experiment on yellow fever.
    h. Faraday's experiment with a wire in a magnetic field.

14. What is the logical significance of the guinea pig?

Chapter Seven

1. How is it possible to believe that nature is not uniform and yet conduct significant experiments?
2. Using a concrete situation in which the familiar concept of cause appears, analyze the meaning of "cause." What difficulties do you find in the concept?
   Describe the problem that would face you if you lived in a world in which there were no causal connections.
3. In what ways do the meanings of the three major assumptions of the experimental sciences overlap one another?
4. How would you proceed by way of analogy to argue that:
   a. a watch which you have just bought will be marked on the inside with a serial number.
   b. you possess an appendix.
   c. there are other solar systems in the universe.
   d. Hitler will be defeated by his own ambition.
   e. western civilization is declining.

   Which of these problems might also be handled effectively by one or more of the inductive methods? Compare the two approaches.
   Which can be argued only on the basis of analogy? Why?
5. Can you cite further cases which may be handled only by way of the argument from analogy?
6. It is generally said that the argument by analogy is one of the most dangerous because most apt to be invalid. Illustrate this from newspaper or radio material.
   Why, then, is it recognized as an important type of argument in the field of the experimental sciences?
7. Taking a specific example from the history of physics, chemistry, botany or zoölogy, illustrate the similarities between building a science and completing a picture puzzle.

8. Using the problem of the derivation of the Table of Atomic Weights, show the relation between inductive and deductive procedures in the field of science.
9. In the case of the Michelson-Morley experiment, a single experiment was deemed sufficient to overthrow the entire theory of ether-drift. Explain.
10. How would you describe the logical difference between the experiments of a seventeenth century scientist and those of a twentieth century scientist?
11. Look up the history of geology and show how the data known in its early stages were organized into more than one system of explanation. What happened, logically, as more data were brought to light?
12. At present the scientific data at our disposal can be organized into more than one system. The following are interesting cases:

    a. expanding vs. contracting universe in astronomy.
    b. mechanism vs. vitalism in biology.
    c. wave theory vs. corpuscular theory of matter in physics.

    How do these conflicts illustrate the logic of science?
13. It is said that one scientific system may be more *adequate* and *convenient* than another, but that neither is more *true* than the other. Apply this principle to the conflict between:

    a. the Ptolemaic and the Copernican systems.
    b. mechanism and vitalism.

    What is the logic of this point of view?
14. State concisely the various limitations imposed by logic upon the contributions of experimental science to human knowledge.
15. What does the logic of science tell us about the function of science?
16. In what sense is "nature" an intellectual tool?

Chapter Eight

1. Show how contract bridge might be transformed into a word game by the employment of cards with letters of the alphabet on them. Try playing such a word-bridge.
2. Can you think of a game based on a structure other than arithmetic or the alphabet which was nevertheless developed primarily for its usefulness?
3. Using the board and counters of checkers, work out a new set of rules and thereby a new game.
4. Using the checkerboard, invent a new set of counters and a new set of rules for an interesting game.
5. Using the chess counters, invent a new board and a new set of rules for an interesting game.
6. Invent a playable game which shall employ a unique board, unique counters and unique rules.
7. Analyze the structures of the following games:

    a. backgammon
    b. handball
    c. seven-and-a-half
    d. roulette

Chapter Nine

1. We have cited the fact that the digits in multiples of three always add up to a number divisible by three. You are also aware of the interesting pattern found in the first ten multiples of nine. What other interesting patterns can you find in the system of arithmetic?
2. What pattern can you find in the graphic properties of the more familiar second degree equations in two unknowns?

3. Construct your own magic square using the numbers from one through sixteen. What is its pattern?
4. Analyze trigonometry as a mathematical game.
5. Show the usefulness of the zero as part of the game of arithmetic.
6. Compare as carefully as possible the structures of the roman and the arabic number systems.
7. Make up your own number system. Use it!
8. Make up your own set of postulates for a new non-euclidean plane geometry. What would be some of the theorems in such a geometry?
9. Make all of the comparisons you can between learning to play a new game and learning to play geometry.
10. Mathematical brain teasers are brain teasers because of unusual patterns which they involve. Look up some familiar ones and show that this is the case.
11. Alter the version of plane geometry with which you are familiar by raising an appropriate early theorem to the status of a postulate; then show that one of the former postulates may be considered as a theorem. Prove this new theorem in terms of the new postulate.
12. Devise a new but simple relationship among numbers, invent a symbol for it, and incorporate it into arithmetic. How useful is it? Why has it not been employed before?

Chapter Ten

1. Express the following in words:

a. $p \supset q . p \supset r : \supset : p . \supset . q . r$

b. $p \supset q . r \supset q : \supset : p \vee r . \supset . q$

c. $p \supset q . \equiv : p . \equiv . p . q$

d. $r \supset -p : p . \equiv . q \vee r : \supset : p . -q . \equiv . r$

2. Express the following in symbols:

   a. If the truth of p and q implies the truth of r, then if p is true q implies r.
   b. If p implies q and r implies s, and also either p is true or r is true; then either q is true or s is true.
   c. If it is false that either p is true or q is true, then p is false and q is false.
   d. p implies q when and only when the truth of p is equivalent to the truth of p and q.

3. What is the difference between $p \vee (q \supset r)$ and $(p \vee q) \supset r$? Illustrate.

4. Employing only definitions, rules, and previous theorems, assign reasons for the steps in the proofs of the following theorems:

   a. *Theorem 7*

$$p \vee p . \supset . p .: \supset \because p . \supset . p \vee p : \supset . p \supset p$$
$$p \vee p . \supset . p$$
$$p . \supset . p \vee p : \supset . p \supset p$$
$$p . \supset . p \vee p$$
$$p \supset p$$

Q.E.D.

   b. *Theorem 9*

$$-p \vee p . \supset . p \vee -p$$
$$-p \vee p$$
$$p \vee -p$$

Q.E.D.

5. Prove the following theorems by the methods indicated:

   a. *Theorem 10*, using theorem 9 and Definition A.
   b. *Theorem 19*, using theorems 25 and 5.

6. Prove the following theorems:

   a. $q . \supset . p \supset q$
   b. $p \supset q . \supset . -q \supset -p$

7. Examine the following theorems. Do they belong to the system of propositions? Why?

a. $p \supset q . \supset . -p \supset -q$
b. $p . \supset : q . \supset . p . q$
c. $p : v : p \vee q . \supset . q$
d. $p \supset q . \supset . q \supset p$
e. $p \vee q . \supset . p \supset -q$
f. $-p \supset p . \supset . p$

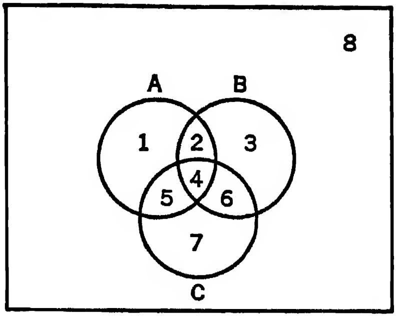
8. Work out the elementary details of a system of propositions which employs:

"v" as "either . . . or, but not both"
and "$\supset$" as "if and only if"

9. Find for yourself details in which the structures of algebra and the system of propositions are alike.
10. Describe in your own words the essential difference between the structures of algebra and the system of propositions.
11. Work out a system of propositions such as might be employed by inhabitants of Saturn. State and prove two of the early theorems in this system. Comment.
12. Take *any* set of primitive symbols, establish by means of additional symbols *any* definitions, work out an arbitrary set of rules. Examine the resulting system for possible meaning and application.

Chapter Eleven

1. Express the following areas in letters:



a. $1 + 2 + 3 + 4 + 5 + 6 + 7$
b. $3 + 4 + 5 + 6$
c. $1 + 2 + 7$
d. $5 + 8$
e. $1 + 6 + 8$

2. Express the following areas in numbers:

a. $A + B$
b. $B \times C$
c. $A + (B \times C)$
d. $(A + B) \times C$
e. $(A + B) \times 1$
f. $(A + B) + 1$
g. $(A \times B \times C) + 0$
h. $(A \times B \times C) \times 0$

3. Express the following in words and give concrete illustrations:

1. $a < b$ is equivalent to $-b < -a$
2. If $a \times -b = 0$ and $b \times -c = 0$, then $a \times -c = 0$
3. $a < 1$
4. $a + b = a + (b \times -a)$

4. Express the following in symbols:

1. Things which are both apples and fruit are included in things which are either apples or fruit.

2. If one class is identical with another, then things which are either in the first class or a third class are identical with things which are either in the second class or the third.
3. Things which are either corkscrews or anything, are any things.
4. If the class of things which are both Fords and not automobiles is a null class, then things which are both Fords and automobiles are Fords.

5. Give reasons for the steps in the following proofs:

1. $a\,(a + -a) = a$
$(a \times a) + (a \times -a) = a$
$(a \times a) + 0 = a$
$a \times a = a$

Q.E.D.

2. $(a \times 0) + 0 = a \times 0$
$0 + (a \times 0) = a \times 0$
$(a \times -a) + (a \times 0) = a \times 0$
$a \times (-a + 0) = a \times 0$
$a \times -a = a \times 0$
$0 = a \times 0$

Q.E.D.

6. Prove the following theorems using the methods indicated:

1. $a < a$, using Rule III and Definition A.
2. *If* $a \times -b = O$, *then* $a \times b = a$, using Rule II, Rule III, Rule VII, the hypothesis that $a \times -b = 0$, and Rule I, in that order.

7. Prove the following:

1. Theorem 15, using three rules.
2. Theorem 25, using a definition and previous theorems.
3. Theorem 29, using Rule V and previous theorems.

8. State the theorems that would be paired with the following ones:

1. $a + (b + c) = (a + b) + c$
2. $a \times -b = 0$ is equivalent to $a + b = b$
3. $a + (a \times b) = a$
4. If $a \times b = 0$, then $b \times a = 0$

9. Examine the following theorems. Do they belong to the system of classes? Why?

 1. If $a < b$ and $a < c$, then $a < b \times c$
 2. If $a < c$ and $b < c$, then $a < b + c$
 3. $a + b = a \times (b + -a)$
 4. $a + (a \times b) = 1$
 5. $a < b$ and $b < a$ is equivalent to $a = b$

10. Interpret the following theorems as applying to the Square of Opposition:

 1. If $a \neq 0$ and $a \times -b = 0$, then $a \times b \neq 0$
 2. If $a \times -b = 0$ and $a \times b = 0$, then $a = 0$
 3. If $a \neq 0$, then $a \times b \neq 0$ or $a \times -b \neq 0$

11. Show symbolically that A and O, and E and I are contradictory.
12. Show that Theorem 21 describes the structurc of the syllogism named Baroco, and that Theorem 22 describes Fesapo.
13. In your own words compare the M-S-P symbolism of atomic analysis with the a-b-c symbolism.
14. Find for yourself details in which the structures of algebra and the system of classes are alike.
15. Compare the structures symbolized by the following pairs of relations.

Propositions		*Classes*
$\supset$	and	$<$
v	and	$+$
.	and	$\times$

16. Derive from the following theorems in the system of

classes theorems of the system of propositions which exhibit the same structure.

1. $a \times -b = 0$ is equivalent to $b + -a = 1$
2. $a + b = -(-a \times -b)$
3. If $a < b$ and $a < c$, then $a < b + c$

17. Discuss as fully as you can the question of which of the two logical systems is more fundamental, that of propositions or that of classes.
    Reconsider the whole question when you have read Chapter Twelve.
18. In the light of the material in Chapters Ten and Eleven, what specific answer would you give to the question, "What is logic?"?

Chapter Twelve

1. Set down a list of ten decisions which you have made in the last week.
   On the same sheet of paper write down the judgment or judgments in terms of which you came to each of these decisions.
   Arrange these judgments in a deductive pattern which will include the more fundamental theorems on which they are based.
   Show as fully as possible the deductive character of your thinking in the following fields:

   a. aesthetics
   b. ethics
   c. theory of knowledge
   d. metaphysics

3. Indicate as clearly as you are able the interrelations of your basic judgments in each of the four fields cited above.

Do you find any contradictions? What is their significance to you?

4. Set down as well as you understand them the basic rules according to which you play the game of life.
   Compare your list with that of some one else. Is there a possibility of argument between you? If so, what does that indicate?
5. Cite instances in your own thinking when you have:
   a. not gone below the surface of your thinking.
   b. gone below the surface but not recognized inconsistencies.
   c. gone below the surface and recognized the more apparent inconsistencies but failed to coördinate your thinking in one field with your thinking in another.

   Show the limitations which these failures have imposed upon your conduct.
6. List twenty familiar ethical precepts. Show which might belong to a single system and which must be classified under different systems. Characterize in terms of fundamental rules the various systems encountered in this search.

Chapter Thirteen

1. Illustrate from your own experience the difference between static and dynamic thinking.
2. Show clearly in what sense the two following arguments are circular:
   a. All sensible men spend their money wisely. We know, then, that none who insist on dressing in style are sensible, because no one who spends his money wisely will insist on dressing in style.
   b. If he did what you say he did he must be either malicious or ignorant. I am forced to the conclusion that he is malicious, because I know that he is not ignorant.

3. Enumerate some of the details of the structure of thinking which have been discovered:
   a. by means of still "photographs"
   b. by means of moving "pictures"

   What comparisons can you draw between the two lists? Why had these things not been discovered before?
4. Find in a novel or play an example of the process of maturing thought.
5. Express in your own words and from your own experience the inadequacies of atomic and molecular analyses of thinking.

Chapter Fourteen

1. Write a Socratic dialogue which will show clearly the importance of doubting.
2. Give a detailed example from your own experience of maturing thought and show how it fulfils each of the five principles of the maturing process.
3. Using the five principles as a basis, write an essay which shall give helpful advice to one seeking maturity of judgment.
4. Can you think of a diagram other than that of the staircase which will illustrate fully and significantly the structure of the maturing process?
5. It might be said that on the second principle rests the entire justification of the logic of maturing thought. In what sense is this true?
6. Cite contradictions in your own thinking which have led to more mature levels of understanding.
   What contradictions still face you, and what is the significance of each?
7. By using concrete examples, give as detailed a description

as possible of what is meant by a synthetic relation in thinking as distinguished from an analytic relation.

8. What changes does the fifth principle of maturing thought make in the ordinary interpretation of man's intellectual activity?
9. Measure as well as you can the maturity of the following: individuals:

Hitler	Martin Luther
St. Paul	Abraham Lincoln
Rousseau	Shakespeare
Erasmus	Casanova
Beethoven	Machiavelli
Cellini	Norman Thomas
Henry Ford	Madame Curie

10. If you were asked to devise a test of mature thinking how would you go about the task?

Chapter Fifteen

1. From what you know of the structure of maturing thought, what prophecy can you make regarding the next one hundred years of philosophic thinking? the next one thousand years?
2. Choosing from the history of philosophy:
   a. the development of the concept of the Good Life from Aristippus to Marcus Aurelius, or
   b. the development of the theory of knowledge from Descartes to Hegel, or
   c. some other development with which you are familiar,

   trace the movement in such a way as to illustrate the major characteristics of the process of maturing thought.
3. Find your own examples in the history of philosophy of

the three types of repetition which it exhibits. Analyze each from the point of view of logic and examine its significance.

4. Which of the phenomena of repetition do you find in your own thinking?
5. Discuss the importance in this new logic of the distinction between the structure and the content of thinking.
6. What implications does the logic of maturing thought have for action in the contemporary world? Treat specifically here the social, economic, political and religious issues with which we are faced.

Table of Immediate Inference

(*see next page*)

Converse:	subject and predicate change places, quality remains same.
Obverse:	negate predicate, and change quality of whole proposition.
Obverted converse:	convert, then obvert.
Partial contrapos:	obvert, then convert.
Full contrapositive:	obvert, convert, then obvert again.
Partial inverse:	*of A*—obvert, convert, obvert, convert, and obvert.
	of E—convert, obvert, and convert.
Full inverse:	*of A*—obvert, convert, obvert, and convert.
	of E—convert, obvert, convert, and obvert.

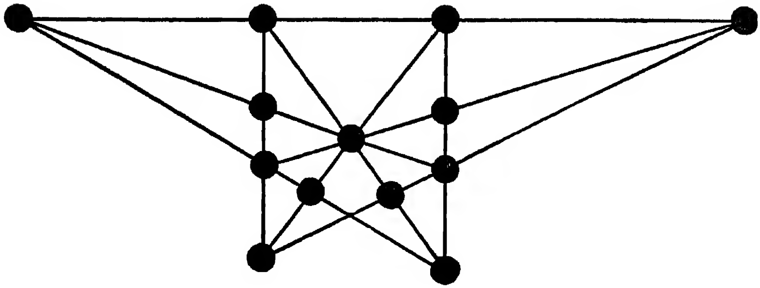
Reading down the table you will find all of the propositions immediately inferable from A, E, I, and O propositions. The parenthesis indicates that the inference is not valid unless "all" and "no" involve the existence of the subject in question.

Table of Immediate Inferences
adapted from
Eaton's *General Logic* (Scribners), p. 211

	A	E	I	O
Original proposition	All S is P	No S is P	Some S is P	Some S is not P
Converse	I (Some P is S)	E No P is S	I Some P is S	
Obverse	E No S is not-P	A All S is not-P	O Some S is not not-P	I Some S is not-P
Obverted converse	O (Some P is not not-S)	A All P is not-S	O Some P is not not-S	
Partial contra-positive	E No not-P is S	I (Some not-P is S)		I Some not-P is S
Full con-trapositive	A All not-P is not-S	O (Some not-P is not not-S)		O Some not-P is not not-S
Partial inverse	O (Some not-S is not P)	I (Some not-S is P)		
Full inverse	I (Some not-S is not-P)	O (Some not-S is not not-P)		

ANSWERS TO BRAIN TEASERS

1.



2. She is 42, and he is 56.

3. Logic is the art of going wrong with confidence.

4.

$$\begin{array}{r} 9567 \\ +1085 \\ \hline \$106.52 \end{array}$$

5. If the squares are numbered from left to right the following squares will be vacant in succession: 4, 3, 5, 6, 4, 2, 1, 3, 5, 7, 6, 4, 2, 3, 5, 4.

6. 7¢ each, or $1.75.

7. Number the girls 1, 2, and 3; and their fiancés, 4, 5, and 6, respectively.

1 and 2 rode to the movie house, and 2 came back.
2 and 3 " " " " ", and 3 " "
4 and 5 " " " " ", and 2 came back.
5 and 6 " " " " ", and 1 came back.
1 and 2 " " " " ", and 6 " "
3 and 6 " " " " "

8. Fifteen ships.

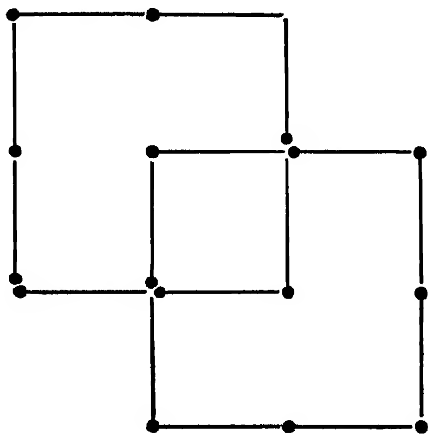
9. Two brave men and a coward.

10. 60 miles.

11. Take a six inch cube, cut it into a cylinder six inches high and six inches in diameter, then from the cylinder make a wedge six inches high with a six inch edge and a circular base six inches in diameter. The wedge is a circle at its base, has a square cross-section through its edge and a triangular cross-section at right angles to the edge.

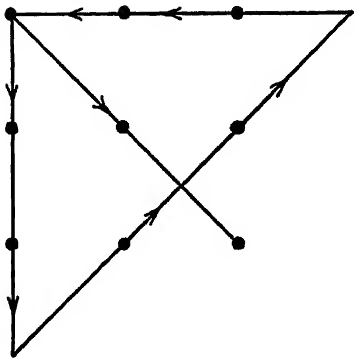
12. Fill the 5 and 11 gill containers from the samovar.
Pour the remaining 8 gills into the 13 gill container.
Put the 5 and 11 gills back into the samovar.
Pour the 8 gills from the 13 into the 11 gill container.
Fill the 13 gill container from the samovar.
Fill the 5 gill container from the 13 gill one.
Pour the 5 gills back into the samovar.

13.



14. 3 m.p.h.

15.

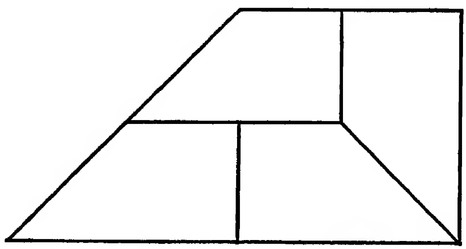


16. They are the same.

17. Five.

18. Five men, one woman, and ninety-four children.

19.



20. 55 minutes.

21. Engine pushes A to the bumper, then comes back to main track and pushes B to A. B and A are joined and the train taken to the main track, where A and B are uncoupled and B pushed to the bumper. Then the engine takes A to the right siding, goes back by way of the main track, and pulls B to the left siding.

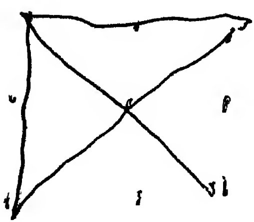
22. Two hours.

23. The man realized that *if* he did *not* have a smudge on his forehead one of the other two, seeing only *one* smudge and all *three* laughing, would immediately know he had a smudge and wipe it off. Neither of the other two did this. Therefore, the first man must have had a smudged forehead.

24. Ten dollars.

25. Brown.

D. for D.D.'s Through twenty-one.



A LOGIC BIBLIOGRAPHY

A. Books of Historical Importance

Aristotle—*Prior Analytics*
——— *Posterior Analytics*
——— *Topics*
——— *Sophistic Elenchi*
Bacon—*Novum Organon*
Arnauld—*The Port-Royal Logic*
Hegel—*Wissenschaft der Logik.*
——— *Encyclopädie der philosophischen Wissenschaften*
J. S. Mill—*A System of Logic*
Boole—*The Mathematical Analysis of Logic*
——— *An Investigation of the Laws of Thought*
de Morgan—*Formal Logic*
Schroeder—*Der Operationskreis des Logikkalkuls*
Frege—*Grundgesetze der Arithmetik*
Peano—*Formulaire de Mathématiques*
Whitehead and Russell—*Principia Mathematica*
Croce—*Logica come scienza del concetto puro*
Gentile—*Sistema di logica come teoria del conoscere*
Wittgenstein—*Tractatus Logico-philosophicus*

B. Books Introductory in Character

Burtt—*Principles and Problems of Right Thinking*
Castell—*A College Logic*
Chapman and Henle—*The Fundamentals of Logic*
Cohen and Nagel—*An Introduction to Logic and Scientific Method*

Dewey—*How We Think*
Eaton—*General Logic*
Evans and Gambertsfelder—*Logic*
Jevons—*Elementary Lessons in Logic*
Joseph—*An Introduction to Logic*
Patterson—*Principles of Correct Thinking*
Robinson—*The Principles of Reasoning*
Stebbing—*A Modern Introduction to Logic*
Ushenko—*The Theory of Logic*

C. Books More Advanced

Bosanquet—*Logic*
Bradley—*The Principles of Logic*
Bridgman—*The Logic of Modern Science*
Dewey—*Essays in Experimental Logic*
——— *Studies in Logical Theory*
——— *Logic*
Eaton—*Symbolism and Truth*
Hobhouse—*The Theory of Knowledge*
Holmes—*The Idealism of Giovanni Gentile. Part Two*
Jevons—*The Principles of Science*
Johnson—*Logic. Three Parts*
Jørgensen—*Treatise of Formal Logic*
Keynes—*Formal Logic*
——— *A Treatise on Probability*
Lewis—*A Survey of Symbolic Logic*
Lewis and Langford—*Symbolic Logic*
Nicod—*La Problème Logique de l'Induction*
Pierce—*Chance, Love and Logic*
——— *Studies in Logic*
Poincaré—*Science and Method*
——— *Science and Hypothesis*
Russell—*Principles of Mathematics*

——— *Introduction to Mathematical Philosophy*
Schiller—*Formal Logic*
——— *Logic for Use*
Venn—*Symbolic Logic*
Wallace—*The Logic of Hegel* (trans. and notes)
Whitehead—*Symbolism*
——— *Introduction to Mathematics*

D. Articles

Holmes—"Classical and Relational Logic" (*Philosophical Review*, Vol. LXV)
Huntington—"Sets of Independent Postulates for the Algebra of Logic" (*Transc. Amer. Math. Soc.*, Vol. 5)
Ladd-Franklin—"On the Algebra of Logic" (Johns Hopkins *Studies in Logic*)
——— "Symbolic Logic" (Baldwin's *Dictionary of Philosophy*)
Royce—"The Principles of Logic" (*Encyclopedia of the Philosophical Sciences*, Vol. 1)
Russell—"The Theory of Implication" (*American Journal of Mathematics*)
——— "Mathematical Logic as based on the Theory of Types" (*American Journal of Mathematics*)
——— "The Philosophical Importance of Mathematical Logic" (*Monist*, Vol. 23)
Sheffer—"Review of *Principia Mathematica*" (*Isis*, No. 25)
——— "A Set of Five Independent Postulates for Boolean Algebra" (*Transc. Amer. Math. Soc.*, Vol. 14)

E. Books General in Character

Ball—*Mathematical Recreations and Essays*
Cohen—*Reason and Nature*
Descartes—*Discours de la Methode*
Dimnet—*The Art of Thinking*
Hocking—*Types of Philosophy*
Joachim—*The Nature of Truth*
Kohn—*Force or Reason*
Lewis—*Mind and the World Order*
Perry—*The Philosophy of the Recent Past*
Plato—*Apology*
Rogers—*The History of Philosophy*
Russell—*Mysticism and Logic*
Santayana—*The Life of Reason*
Spinoza—*Ethics*
Whitehead—*The Concept of Nature*
——— *The Function of Reason*
Young—*The Fundamental Concepts of Algebra and Geometry*

INDEX

(1)